

A class of asymmetric gapped Hamiltonians on quantum spin chains and its characterization II

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Abstract

We give a characterization of the class of gapped Hamiltonians introduced in Part I [O]. The Hamiltonians in this class are given as MPS (Matrix product state) Hamiltonians. In [O], we list up properties of ground state structures of Hamiltonians in this class. In this Part II, we show the converse. Namely, if a (not necessarily MPS) Hamiltonian H satisfies five of the listed properties, there is a Hamiltonian H' from the class in [O], satisfying the followings: The ground state spaces of the two Hamiltonians on the infinite intervals coincide. The spectral projections onto the ground state space of H on each finite intervals are approximated by that of H' exponentially well, with respect to the interval size. The latter property has an application to the classification problem with open boundary conditions.

1 Introduction

In Part I [O], we introduced a class of MPS Hamiltonians, which allows asymmetric ground state structures. There, we gave a list of physical properties that the ground states of these Hamiltonians satisfy. In this Part II, conversely, we show that these physical properties actually guarantee the ground state structure of the Hamiltonian to be captured by the class of Hamiltonians we introduced. More precisely, we will show if a Hamiltonian satisfies five physical conditions, there is an MPS Hamiltonian from our class satisfying the followings: 1. The ground state spaces of the two Hamiltonians on the infinite intervals coincide. 2. The spectral projections onto the ground state space of the original Hamiltonian on finite intervals are well approximated by that of the MPS one. From the latter property we see that two Hamiltonians are in the same class in the classification problem of gapped Hamiltonians.

We use freely the notations and definitions given in Part I, Subsection 1.1, 1.2, 1.3, and Appendix A. In particular, recall the definition of ClassA. In Part I, we studied the properties of ground state structures of MPS Hamiltonians given by elements in ClassA. We consider the quantum spin chain described in Subsection 1.1 of Part I. Let $n \in \mathbb{N}$ with $n \geq 2$.

Assumption 1.1. Let $m \in \mathbb{N}$ and h a positive element in $\mathcal{A}_{[0, m-1]}$, and let H be the Hamiltonian given by h via the formula (1) and (2) of Part I. We consider the following conditions.

A1 There exist $N_1, d_1 \in \mathbb{N}$ such that $1 \leq \dim \ker(H)_{[0, N-1]} \leq d_1$ for all $N_1 \leq N \in \mathbb{N}$.

A2 Let G_N be the orthogonal projection onto $\ker(H)_{[0, N-1]}$ acting on $\bigotimes_{i=0}^{N-1} \mathbb{C}^n$. There exist $\gamma > 0$ and $N_2 \in \mathbb{N}$ such that

$$\gamma(1 - G_N) \leq (H)_{[0, N-1]}, \text{ for all } N \geq N_2.$$

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A3 $\mathcal{S}_{\mathbb{Z}}(H)$ consists of a unique state ω_{∞} on $\mathcal{A}_{\mathbb{Z}}$,

A4 There exist $0 < C_1$, $0 < s_1 < 1$, $N_3 \in \mathbb{N}$ and factor states $\omega_R \in \mathcal{S}_{[0,\infty)}(H)$, $\omega_L \in \mathcal{S}_{(-\infty,-1]}(H)$, such that

$$\begin{aligned} \left| \frac{\text{Tr}_{[0,N-1]}(G_N A)}{\text{Tr}_{[0,N-1]}(G_N)} - \omega_R(A) \right| &\leq C_1 s_1^{N-l} \|A\|, \\ \left| \frac{\text{Tr}_{[0,N-1]}(G_N \tau_{N-l}(A))}{\text{Tr}_{[0,N-1]}(G_N)} - \omega_L \circ \tau_{-l}(A) \right| &\leq C_1 s_1^{N-l} \|A\|, \end{aligned} \quad (1)$$

for all $l \in \mathbb{N}$, $A \in \mathcal{A}_{[0,l-1]}$, and $N \geq \max\{l, N_3\}$, and

$$\begin{aligned} \inf \left\{ \sigma \left(D_{\omega_R|_{\mathcal{A}_{[0,l-1]}}} \right) \setminus \{0\} \mid l \in \mathbb{N} \right\} &> 0, \\ \inf \left\{ \sigma \left(D_{\omega_L|_{\mathcal{A}_{[-l,-1]}}} \right) \setminus \{0\} \mid l \in \mathbb{N} \right\} &> 0. \end{aligned} \quad (2)$$

A5 For any $\psi \in \mathcal{S}_{[0,\infty)}(H)$ (resp. $\psi \in \mathcal{S}_{(-\infty,-1]}(H)$), there exists an $l_{\psi} \in \mathbb{N}$ such that

$$\|\psi - \psi \circ \tau_{l_{\psi}}\| < 2, \quad (\text{resp. } \|\psi - \psi \circ \tau_{-l_{\psi}}\| < 2).$$

The main Theorem of Part II is the following.

Theorem 1.2. *Let $n \in \mathbb{N}$ with $n \geq 2$. Let $m \in \mathbb{N}$, and h a positive element in $\mathcal{A}_{[0,m-1]}$. Let H be the Hamiltonian given by this h . Assume that [A1]-[A5] hold for this h . Then there exist $\mathbb{B} \in \text{ClassA}$ and $m_1 \in \mathbb{N}$ such that*

$$\mathcal{S}_{(-\infty,-1]}(H) = \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m_1,\mathbb{B}}}), \quad \mathcal{S}_{[0,\infty)}(H) = \mathcal{S}_{[0,\infty)}(H_{\Phi_{m_1,\mathbb{B}}}), \quad \mathcal{S}_{\mathbb{Z}}(H) = \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m_1,\mathbb{B}}}). \quad (3)$$

Furthermore, there exist $C > 0$, $0 < s < 1$, and $N_0 \in \mathbb{N}$ such that

$$\|G_N - G_{N,\mathbb{B}}\| \leq C s^N, \quad N \geq N_0. \quad (4)$$

If \mathbb{B} belongs to ClassA with respect to $(n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$, the above m_1 satisfies

$$m_1 \geq \max \left\{ 2l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y), \frac{\log(n_0^2(k_L + 1)(k_R + 1) + 1)}{\log n} \right\}.$$

The condition [A1] means that the Hamiltonian is frustration free, and its ground state dimensions on finite intervals are uniformly bounded. This is inevitable if one hopes to describe the ground state structure in terms of matrices. The second condition [A2] means that the Hamiltonian is gapped. This is the condition under which we would like to work. The third one [A3] means that the bulk ground state is unique. We should be able to extend the Theorem to the case that some discrete symmetry is broken in the bulk, but for the simplicity, we assume the uniqueness in the current paper. The equations (1) in [A4] can be called the edge versions of (a relaxed) local topological quantum order introduced in [MP]. It says that the effect of the edge away from the place the observation is made, decays exponentially fast with respect to the distance. In [MP], local topological quantum order was assumed to guarantee the stability of the spectral gap. The equations (2) requires the non-existence of the zero-mode. The requirement that ω_R, ω_L are factor states can be understood that these states are in pure phase [BR2]. [A5] is rather a technical condition. Recall that the maximal distance that two states can have is 2. This condition [A5] says that for any edge ground state, there exists a space translation of it whose distance from the original one is smaller than this maximal number 2.

From Theorem 1.18 of Part I, $H_{\Phi_{m_1,\mathbb{B}}}$ with $\mathbb{B} \in \text{ClassA}$ and m_1 large enough, satisfies [A1]-[A5]. In this sense, [A1]-[A5] can be understand as a qualitative characterization of ClassA.

We would like to emphasize that in this approximation (4), the size of the matrices \mathbb{B} is fixed, i.e., it does not grow with respect to the size of the interval $[0, N-1]$, nor the precision of the approximation. The existence of n -tuple of matrices which describes the bulk ground state of Hamiltonians satisfying [A1] is known [M1], [M2]. The question here is how to deal with the edge states.

Theorem 1.2 has an application to the classification problem of gapped Hamiltonians. Here, we would like to extend the notion of C^1 -classification of gapped Hamiltonians with open boundary conditions considered in [BO] (which we would like to call C^1 -classification I of gapped Hamiltonians with open boundary conditions, to distinguish from the following one). We use the notations and definitions in [BO].

Definition 1.3 (C^1 -classification II of gapped Hamiltonians with open boundary conditions). Let H_0, H_1 be gapped Hamiltonians associated with interactions $\Phi_0, \Phi_1 \in \mathcal{J}$. We say that H_0, H_1 are C^1 -equivalent of type II if the following conditions are satisfied.

1. There exist $m \in \mathbb{N}$ and a continuous and piecewise C^1 -path $\Phi : [0, 1] \rightarrow \mathcal{J}_m$ such that $\Phi(0) = \Phi_0$, $\Phi(1) = \Phi_1$. Let $H(t)$ be the Hamiltonian associated with $\Phi(t)$ for each $t \in [0, 1]$.
2. There are $\gamma > 0$, $N_0 \in \mathbb{N}$, and finite intervals $I(t) = [a(t), b(t)]$, $t \in [0, 1]$, satisfying the followings:
 - (i) the endpoints $a(t), b(t)$ smoothly depends on $t \in [0, 1]$,
 - (ii) there exists a sequence $\{\varepsilon_N\}_{N \in \mathbb{N}}$ of positive numbers with $\varepsilon_N \rightarrow 0$, for $N \rightarrow \infty$, such that $\sigma(H(t)_{[0, N-1]}) \cap I(t) = \sigma(H(t)_{[0, N-1]}) \cap [\lambda(t, N), \lambda(t, N) + \varepsilon_N]$, and $\sigma(H(t)_{[0, N-1]}) \cap I(t)^c = \sigma(H(t)_{[0, N-1]}) \cap [b(t) + \gamma, \infty)$ for all $N \geq N_0$ and $t \in [0, 1]$, where $\lambda(t, N)$ is the smallest eigenvalue of $H(t)_{[0, N-1]}$.

From Theorem 1.2, we obtain the following:

Corollary 1.4. Let $n \in \mathbb{N}$ with $n \geq 2$. Let $m \in \mathbb{N}$, and h a positive element in $\mathcal{A}_{[0, m-1]}$. Let H be the Hamiltonian given by this h . Assume that [A1]-[A5] hold for this h . Let $\mathbb{B} \in \text{ClassA}$ and $m_1 \in \mathbb{N}$ given in Theorem 1.2 for this h . Then there exist $\hat{\gamma} > 0$, $0 < c_1$, $0 < s_1, s_2 < 1$, and $\hat{N}_0 \in \mathbb{N}$ such that

$$\sigma\left((1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]}\right) \cap [\hat{\gamma}c_1s_1^N, \hat{\gamma} - s_2^N\hat{\gamma}] = \emptyset, \quad t \in [0, 1], \quad N \geq \hat{N}_0.$$

In particular, H and $H_{\Phi_{m_1, \mathbb{B}}}$ are C^1 -equivalent of type II.

Let us call $\tilde{\mathcal{H}}$ the set of all Hamiltonians satisfying the qualitative conditions [A1]-[A5]. The Corollary 1.4 means that for the C^1 -classification of type II, each equivalence class with an element from $\tilde{\mathcal{H}}$ includes an element $H_{\Phi_{m_1, \mathbb{B}}}$ which is given by some $\mathbb{B} \in \text{ClassA}$. Therefore, if we consider only $\tilde{\mathcal{H}}$, it suffices to consider MPS Hamiltonians given by $\mathbb{B} \in \text{ClassA}$. As ClassA is given by quite concrete conditions, this is an advantage.

This article is organized as follows. In Section 2, we give representations of $\mathcal{S}_{(-\infty, -1]}(H)$, and $\mathcal{S}_{[0, \infty)}(H)$ by matrices. They are given by different n -tuples of matrices $\mathbf{v}_L, \mathbf{v}_R$. In our setting, from the theorem of Arveson [A], we can find a representation of the Cuntz algebra associated to the space translation [BJ]. The argument of Matsui [M1][M2] then shows that from the representation, we can find an n -tuple of matrices which describes the bulk ground state. In Section 2, we overview these materials. We then start to investigate the structure of these matrices. In Section 3, we find that the upper triangular property emerges by the hierarchical structure of \mathbf{v}_σ^* -invariant subspaces $\mathcal{K}_{0\sigma} \subsetneq \mathcal{K}_{1\sigma} \subsetneq \cdots \subsetneq \mathcal{K}_{b\sigma}$, (Lemma 3.4). We denote by $p_{a\sigma}$ the orthogonal projection onto $\mathcal{K}_{a\sigma}$ and set $r_{a\sigma} := p_{a\sigma} - p_{a-1, \sigma}$ and $\omega_{i, a, \sigma} = r_{a\sigma} v_{i\sigma} r_{a\sigma}$. Each $T_{\omega_{a\sigma}}$ is nonzero due to [A5] (Lemma 3.8), and is an irreducible CP map (Lemma 3.4). In Section 4, we show that this map (or $T_{\mathbf{u}_{a\sigma}}$ which is similar to it) is primitive (Lemma 4.1). This follows from the fact that this $\omega_{a\sigma}$ generates an

element of $\mathcal{S}_{\mathbb{Z}}(H)$, i.e., from [A3], it generates ω_{∞} (Lemma 4.2). Applying a theorem from [FNW2], we obtain the primitivity. More precisely, $\mathbf{u}_{a\sigma}$ are unitarily equivalent to the primitive n -tuple of matrices which gives the minimal representation of ω_{∞} . From this observation, we obtain the structure $v_{\mu\sigma} \in M_{n_0} \otimes M_{k_{\sigma}+1}$. Next, we investigate the $M_{k_{\sigma}+1}$ part. The condition (2) in [A4] implies $\Gamma_{l, \mathbf{v}_{\sigma}}^{(\sigma)} \Big|_{B(\mathcal{K}_{\sigma})r_{0\sigma}}$ is an injection onto $\Gamma_{l, \mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma}))$, for l large enough (Lemma 5.1). From this fact, we obtain a basis of $\mathcal{K}_l(\mathbf{v}^{(\sigma)})$, $\{y_{a, \alpha, \beta}^{(l, \sigma)}\}$ satisfying the conditions in Lemma 6.4. It turns out that these conditions are so restrictive that we obtain $\lambda, \mathbb{D}, \mathbb{G}, Y, \{\hat{x}_{\mu, b}^{(L)}\}$ and $\{\hat{x}_{\mu, a}^{(R)}\}$ (Lemma 6.10). The key for this procedure is Lemma 6.6. Out of these objects we obtain by the end of Section 6, we construct $\mathbb{B} \in \text{Class } A$ in Section 7. This \mathbb{B} is obtained by embedding M_{k_L+1} and M_{k_R+1} into $M_{k_L+k_R+1}$. It satisfies $\mathcal{S}_{[0, \infty)}(H_{\Phi_{m', \mathbb{B}}}) = \mathcal{S}_{[0, \infty)}(H)$, $\mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m', \mathbb{B}}}) = \mathcal{S}_{(-\infty, -1]}(H)$, and $\omega_{\infty} = \omega_{\mathbb{B}, \infty}$. In the final section, we show that $G_{N, \mathbb{B}}$ s approximate G_N s exponentially well, with respect to N . The spectral gap and the assumption that the effect of the boundary disappears exponentially fast ([A4]) show (4).

Remark 1.5. In addition to the notations given in Subsection 1.1, 1.2, 1.3, and Appendix A of Part I, we use the notations given in Appendix A.

2 Representation of $\mathcal{S}_{(-\infty, -1]}(H)$, $\mathcal{S}_{[0, \infty)}(H)$ by matrices

In this section, we give a representation of elements in $\mathcal{S}_{(-\infty, -1]}(H)$, $\mathcal{S}_{[0, \infty)}(H)$ by matrices. Readers should be aware that at this point, the matrices representing $\mathcal{S}_{(-\infty, -1]}(H)$ and $\mathcal{S}_{[0, \infty)}(H)$ are different. Most of the following Lemmas are well known. (See [BJ][M1][M2])

Lemma 2.1. *Let $\sigma = L, R$. Assume [A1]. Then the followings hold.*

1. *For a state ϱ_{σ} on \mathcal{A}_{σ} , ϱ_{σ} belongs to $\mathcal{S}_{\sigma}(H)$ if and only if $\varrho_{\sigma}(\tau_x(h)) = 0$ for all $x \in \mathbb{Z}^{(\sigma)}$. For a state ϱ on $\mathcal{A}_{\mathbb{Z}}$, ϱ belongs to $\mathcal{S}_{\mathbb{Z}}(H)$ if and only if $\varrho(\tau_x(h)) = 0$ for all $x \in \mathbb{Z}$.*
2. *$\mathcal{S}_{\sigma}(H)$ is a wk^* -compact convex face in the set of all states on \mathcal{A}_{σ} .*
3. *There exists a pure state φ_{σ} in $\mathcal{S}_{\sigma}(H)$.*

Proof. The proof of 1. is the same as the proof of Lemma 3.10 in Part I [O]. 2 is clear from 1. 3 follows from 2 and the Klein-Milman theorem. \square

Remark 2.2. From Lemma 2.1, we choose and fix one pure state φ_{σ} in $\mathcal{S}_{\sigma}(H)$ from now on.

Lemma 2.3. *Let $\sigma = L, R$. Assume [A1] and [A4]. Then we have the followings.*

1. *For $d_1 \in \mathbb{N}$ in [A1] and the state ω_{σ} given in [A4], we have $\varrho_{\sigma} \leq d_1 \cdot \omega_{\sigma}$ for any $\varrho_{\sigma} \in \mathcal{S}_{\sigma}(H)$.*
2. *Any elements in $\mathcal{S}_{\sigma}(H)$ are mutually quasi-equivalent.*

Proof. To prove 1., let $D_{\varrho_R|_{\mathcal{A}_{[0, N-1]}}}$ be the reduced density matrix of $\varrho_R \in \mathcal{S}_R(H)$, on $\mathcal{A}_{[0, N-1]}$. Then, by [A1] and 1. of Lemma 2.1, we have

$$0 \leq \varrho_R(A) = \text{Tr} \left(D_{\varrho_R|_{\mathcal{A}_{[0, N-1]}}} A \right) \leq \text{Tr} G_N A \leq d_1 \frac{\text{Tr} G_N A}{\text{Tr} G_N},$$

for any $l \leq N$ and $A \in \mathcal{A}_{[0, l-1], +}$. Taking $N \rightarrow \infty$ limit, and from [A4], we obtain

$$0 \leq \varrho_R(A) \leq d_1 \omega_R(A),$$

for any $l \in \mathbb{N}$ and $A \in \mathcal{A}_{[0, l-1], +}$, proving 1. for $\sigma = R$. The same argument proves 1. for $\sigma = L$. From 1., any $\varrho_{\sigma} \in \mathcal{S}_{\sigma}(H)$ is ω_{σ} -normal. As ω_{σ} is a factor state, this means that ϱ_{σ} and ω_{σ} are quasi-equivalent, proving 2. \square

The following (except for 6) is the list of Lemmas proven in [M1] and [M2].

Lemma 2.4. *Let $\sigma = L, R$. Assume [A1] and [A4]. Let $\varphi_\sigma \in \mathcal{S}_\sigma(H)$ be the pure state fixed in Remark 2.2, and $(\mathcal{H}_\sigma, \pi_\sigma, \Omega_\sigma)$ its GNS triple. Define the subspace \mathcal{K}_σ by*

$$\mathcal{K}_\sigma := \bigcap_{x \in \mathbb{Z}^{(\sigma)}} \ker \pi_\sigma(\tau_x(h)) \subset \mathcal{H}_\sigma$$

Then the followings hold.

1. $1 \leq \dim \mathcal{K}_\sigma \leq d_1$.
2. φ_σ and $\varphi_\sigma \circ \tau_x^{(\sigma)}$ are quasi-equivalent for all $x \in \mathbb{N}$.
3. There exists $S_{i,\sigma} \in B(\mathcal{H}_\sigma)$, $i = 1, \dots, n$ such that

$$S_{i,\sigma}^* S_{j,\sigma} = \delta_{ij}, \quad \sum_{j=1}^n S_{j,\sigma} \pi_\sigma(A) S_{j,\sigma}^* = \pi_\sigma \circ \tau_1^{(\sigma)}(A), \quad A \in \mathcal{A}_\sigma.$$

with

$$\begin{aligned} \pi_R \left(\bigotimes_{k=0}^{l-1} e_{i_k, j_k}^{(n)} \right) &= S_{(i_0, R)} \cdots S_{(i_{l-1}, R)} S_{(j_{l-1}, R)}^* \cdots S_{(j_0, R)}^*, \quad \text{if } \sigma = R, \\ \pi_L \left(\bigotimes_{k=-l}^{-1} e_{i_k, j_k}^{(n)} \right) &= S_{(i_{-1}, L)} \cdots S_{(i_{-l}, L)} S_{(j_{-l}, L)}^* \cdots S_{(j_{-1}, L)}^*, \quad \text{if } \sigma = L, \end{aligned}$$

for all $l \in \mathbb{N}$, $i_k, j_k \in \{1, \dots, n\}$.

4. For $\{S_{i,\sigma}\}$ in 3, we have $S_{i,\sigma}^* \mathcal{K}_\sigma \subset \mathcal{K}_\sigma$, $i = 1, \dots, n$.
5. There exists one to one correspondence between $\psi \in \mathcal{S}_\sigma(H)$ and a density matrix ρ_ψ in \mathcal{H}_σ with support in \mathcal{K}_σ via

$$\psi(A) = \text{Tr}(\rho_\psi \pi_\sigma(A)), \quad A \in \mathcal{A}_\sigma.$$

In this correspondence, ψ is pure if and only if ρ_ψ is rank one.

6. For ω_σ in [A4], ρ_{ω_σ} is a strictly positive element of $B(\mathcal{K}_\sigma)$.

Proof. Proof of 1 is the same as that of Proposition 5.1 of [M2]. 2. is due to the fact that $\varphi_\sigma \circ \tau_x^{(\sigma)} \in \mathcal{S}_\sigma(H)$ and Lemma 2.3. From 2, we may apply Lemma B.2 to φ_σ and obtain 3,4, as in the proof of Theorem 1.2 [M1]. By Lemma 2.3 2. any $\psi \in \mathcal{S}_\sigma(H)$ is quasi-equivalent to φ_σ . Therefore, it can be represented by a density matrix ρ_ψ on \mathcal{H}_σ . However, as $\psi(\tau_x(h)) = 0$ for all $x \in \mathbb{Z}^{(\sigma)}$, the support of ρ_ψ is in \mathcal{K}_σ . Conversely, if ρ is a density matrix in \mathcal{H}_σ with support in \mathcal{K}_σ , then the state given by $\text{Tr} \rho \pi_\sigma(\cdot)$ belongs to $\mathcal{S}_\sigma(H)$. As φ_σ is pure, we have $\pi_\sigma(\mathcal{A}_\sigma)'' = B(\mathcal{H}_\sigma)$. Therefore, the correspondence above is one to one, and the statement about the purity holds. To prove 6, note that if ρ_{ω_σ} is not strictly positive in $B(\mathcal{K}_\sigma)$, then there exists a unit vector $\xi \in \mathcal{K}_\sigma$ which is orthogonal to $s(\rho_{\omega_\sigma})$. By 5, this ξ defines a state $\omega_\xi = \langle \xi, \pi_\sigma(\cdot) \xi \rangle \in \mathcal{S}_\sigma(H)$. Let p be the orthogonal projection onto ξ . As φ_σ is pure, we have $\pi_\sigma(\mathcal{A}_\sigma)'' = B(\mathcal{K}_\sigma)$. Therefore, by Kaplansky's density Theorem, there exists a net $\{x_\alpha\}_\alpha$ in the unit ball of $\mathcal{A}_{\sigma,+}$, such that $\pi_\sigma(x_\alpha) \rightarrow p$ in the σw -topology. For this net, we have $\lim_\alpha \omega_\sigma(x_\alpha) = 0$ and $\lim_\alpha \omega_\xi(x_\alpha) = 1$. This contradicts $\omega_\xi \leq d_1 \cdot \omega_\sigma$ given in Lemma 2.3. \square

Notation 2.5. Assume [A1] and [A4]. Let $\sigma = L, R$ and φ_σ be the pure state fixed in Remark 2.2. Let \mathcal{K}_σ be the finite dimensional Hilbert space and $\{S_{i,\sigma}\}_{i=1}^n \subset B(\mathcal{H}_\sigma)$ given in Lemma 2.4. By 4 of Lemma 2.4, we can define $v_{i,\sigma} \in B(\mathcal{K}_\sigma)$ by $v_{i,\sigma}^* := S_{i,\sigma}^*|_{\mathcal{K}_\sigma}$, for $i = 1, \dots, n$. We also set $m_\sigma := \dim \mathcal{K}_\sigma$, and define $P_{\mathcal{K}_\sigma}$ to be the orthogonal projection onto \mathcal{K}_σ on \mathcal{H}_σ . We constantly identify $B(\mathcal{K}_\sigma)$, $P_{\mathcal{K}_\sigma} B(\mathcal{H}_\sigma) P_{\mathcal{K}_\sigma}$, and M_{m_σ} .

With this notation, 3,5 of Lemma 2.4 can be rephrased as follows.:

Lemma 2.6. Assume [A1] and [A4]. Let $\sigma = L, R$. Then there exist $m_\sigma \in \mathbb{N}$ and n -tuple of matrices $\mathbf{v}_\sigma = (v_{\mu,\sigma})_{\mu=1}^n \in M_{m_\sigma}^{\times n}$ satisfying the followings.

1.

$$\sum_{j=1}^n v_{j\sigma} v_{j\sigma}^* = \mathbb{I}_{M_{m_\sigma}}.$$

2. There exists one to one correspondence between $\psi \in \mathcal{S}_\sigma(H)$ and a density matrix $\rho_\psi \in M_{m_\sigma}$ via

$$\begin{aligned} \psi \left(\bigotimes_{k=0}^{l-1} e_{i_k, j_k}^{(n)} \right) &= \text{Tr} \left(\rho_\psi \left(v_{(i_0, R)} \cdots v_{(i_{l-1}, R)} v_{(j_{l-1}, R)}^* \cdots v_{(j_0, R)}^* \right) \right), \quad \text{if } \sigma = R, \\ \psi \left(\bigotimes_{k=-l}^{-1} e_{i_k, j_k}^{(n)} \right) &= \text{Tr} \left(\rho_\psi \left(v_{(i_{-1}, L)} \cdots v_{(i_{-l}, L)} v_{(j_{-l}, L)}^* \cdots v_{(j_{-1}, L)}^* \right) \right), \quad \text{if } \sigma = L, \end{aligned}$$

for all $l \in \mathbb{N}$, $i_k, j_k \in \{1, \dots, n\}$. In this correspondence, ψ is pure if and only if ρ_ψ is rank one.

Hence we obtained the n -tuples of matrices $\mathbf{v}_R, \mathbf{v}_L$ which have all the information of $\mathcal{S}_R(H)$ and $\mathcal{S}_L(H)$ respectively. These tuples $\mathbf{v}_R, \mathbf{v}_L$ are not equal in general. The question is how to connect these informations in a way to approximate G_N simultaneously. This requires further investigations on the properties of $\mathbf{v}_R, \mathbf{v}_L$, which will be carried out in the next three sections.

3 A sequence of $v_{i\sigma}^*$ -invariant subspaces

We start from the following observation.

Lemma 3.1. Assume [A1] and [A4]. For $\sigma = L, R$, let $\mathbf{v}_\sigma = (v_{1,\sigma}, \dots, v_{n,\sigma})$ be the matrices given in Notation 2.5. Define for each $N \in \mathbb{N}$,

$$\mathcal{L}_{N,\sigma} := \text{span} \left\{ v_{(i_1, \sigma)} \cdots v_{(i_N, \sigma)} v_{(j_N, \sigma)}^* \cdots v_{(j_1, \sigma)}^* \mid i_k, j_k \in \{1, \dots, n\}, k = 1, \dots, N \right\}. \quad (5)$$

Then we have

$$\mathcal{L}_{1,\sigma} \subset \mathcal{L}_{2,\sigma} \subset \cdots \subset \mathcal{L}_{N,\sigma} \subset \mathcal{L}_{N+1,\sigma} \subset \cdots \subset B(\mathcal{K}_\sigma), \quad (6)$$

and there exists $N_\sigma \in \mathbb{N}$ such that $\mathcal{L}_{N_\sigma, \sigma} = B(\mathcal{K}_\sigma)$. In particular, we have $P_{\mathcal{K}_\sigma} \pi_\sigma(\mathcal{A}_\sigma) P_{\mathcal{K}_\sigma} = B(\mathcal{K}_\sigma)$.

Proof. By Lemma 2.6 1, we have

$$v_{(i_1, \sigma)} \cdots v_{(i_N, \sigma)} v_{(j_N, \sigma)}^* \cdots v_{(j_1, \sigma)}^* = \sum_i v_{(i_1, \sigma)} \cdots v_{(i_N, \sigma)} v_{(i\sigma)} v_{(i\sigma)}^* v_{(j_N, \sigma)}^* \cdots v_{(j_1, \sigma)}^* \in \mathcal{L}_{N+1, \sigma}.$$

This proves the inclusion $\mathcal{L}_{N,\sigma} \subset \mathcal{L}_{N+1,\sigma}$.

Note that by the definition of $v_{i\sigma}$ and Lemma 2.4, we have

$$\bigcup_{N=1}^{\infty} \mathcal{L}_{N,\sigma} = P_{\mathcal{K}_\sigma} \pi_\sigma (\mathcal{A}_\sigma \cap \mathcal{A}_{\mathbb{Z}}^{\text{loc}}) P_{\mathcal{K}_\sigma} \subset B(\mathcal{K}_\sigma).$$

Note that this is a subspace of a finite dimensional vector space $B(\mathcal{K}_\sigma)$. Therefore, it is σw -closed. As φ_σ is pure, we have $\pi_\sigma (\mathcal{A}_\sigma \cap \mathcal{A}_{\mathbb{Z}}^{\text{loc}})'' = B(\mathcal{H}_\sigma)$. Therefore, $\bigcup_{N=1}^{\infty} \mathcal{L}_{N,\sigma}$ is σw -dense in $B(\mathcal{K}_\sigma)$. Hence we obtain $\bigcup_{N=1}^{\infty} \mathcal{L}_{N,\sigma} = B(\mathcal{K}_\sigma)$. Combining this with (6), we conclude that $\mathcal{L}_{N,\sigma} = B(\mathcal{K}_\sigma)$ for N large enough. \square

Lemma 3.2. *Assume [A1], [A3], and [A4]. For $\sigma = L, R$, let $\mathbf{v}_\sigma = (v_{1,\sigma}, \dots, v_{n,\sigma})$ be the matrices given in Notation 2.5. Then followings hold.*

1. $T_{\mathbf{v}_\sigma}$ is a unital CP map such that $\sigma(T_{\mathbf{v}_\sigma}) \cap \mathbb{T} = \{1\}$.
2. There exists a $T_{\mathbf{v}_\sigma}$ -invariant state ρ_σ on M_{m_σ} such that

$$\begin{aligned} P_{\{1\}}^{T_{\mathbf{v}_\sigma}}(\cdot) &= \rho_\sigma(\cdot) \mathbb{I}_{\mathcal{K}_\sigma}, \\ P_{\{1\}}^{T_{\mathbf{v}_\sigma}}(P_{\mathcal{K}_\sigma} \pi_\sigma(A) P_{\mathcal{K}_\sigma}) &= \omega_\infty(A) \mathbb{I}_{\mathcal{K}_\sigma}, \quad A \in \mathcal{A}_\sigma. \end{aligned}$$

Proof. The map $T_{\mathbf{v}_\sigma}$ is a unital CP map because of the definition and Lemma 2.6. In particular, $T_{\mathbf{v}_\sigma}$ is a contraction. Therefore, the spectrum of $T_{\mathbf{v}_\sigma}$ is in the closed unit ball of \mathbb{C} and every Jordan cell of $T_{\mathbf{v}_\sigma}$ corresponding to an eigenvalue in \mathbb{T} has dimension 1. From Lemma 2.4, for any unit vector $\xi \in \mathcal{K}_\sigma$, $\omega_\xi := \langle \xi, \pi_\sigma(\cdot) \xi \rangle$ defines an element in $\mathcal{S}_\sigma(H)$. Choose any state ψ on $\mathcal{A}_{(\Gamma_\sigma)^c}$. Define for each $N \in \mathbb{N}$, a state $\psi_{N,\sigma}$ on $\mathcal{A}_{\mathbb{Z}}$ by $\psi_{N,R} := (\psi \otimes \omega_\xi) \circ \tau_N^{(R)}$ if $\sigma = R$, under the identification $\mathcal{A}_{\mathbb{Z}} \simeq \mathcal{A}_{(\Gamma_R)^c} \otimes \mathcal{A}_R$ and $\psi_{N,L} := (\omega_\xi \otimes \psi) \circ \tau_N^{(L)}$ if $\sigma = L$, under the identification $\mathcal{A}_{\mathbb{Z}} \simeq \mathcal{A}_L \otimes \mathcal{A}_{(\Gamma_L)^c}$. For any $x \in \mathbb{Z}$, we have $\psi_{N,\sigma} \circ \tau_x(h) = 0$ eventually as $N \rightarrow \infty$. Therefore, any wk^* -accumulation point of $\psi_{N,\sigma}$ belongs to $\mathcal{S}_{\mathbb{Z}}(H)$. By [A3], this means that $\psi_{N,\sigma}$ converges to ω_∞ in wk^* -topology as $N \rightarrow \infty$.

For any $A \in \mathcal{A}_\sigma$, by Lemma 2.4, we have

$$\psi_{N,\sigma}(A) = \omega_\xi \circ \tau_N^{(\sigma)}(A) = \left\langle \xi, \pi_\sigma \circ \tau_N^{(\sigma)}(A) \xi \right\rangle = \left\langle \xi, T_{\mathbf{v}_\sigma}^N(P_{\mathcal{K}_\sigma} \pi_\sigma(A) P_{\mathcal{K}_\sigma}) \xi \right\rangle.$$

On the other hand, by the argument above, we have $\lim_N \psi_{N,\sigma}(A) = \omega_\infty(A)$. Therefore, we have

$$\lim_N T_{\mathbf{v}_\sigma}^N(P_{\mathcal{K}_\sigma} \pi_\sigma(A) P_{\mathcal{K}_\sigma}) = \omega_\infty(A) \mathbb{I}_{\mathcal{K}_\sigma}, \quad A \in \mathcal{A}_\sigma. \quad (7)$$

By Lemma 3.1, this means that $\sigma(T_{\mathbf{v}_\sigma}) \cap \mathbb{T} = \{1\}$, $P_{\{1\}}^{T_{\mathbf{v}_\sigma}}(B(\mathcal{K}_\sigma)) = \mathbb{C} \mathbb{I}_{\mathcal{K}_\sigma}$, and

$$P_{\{1\}}^{T_{\mathbf{v}_\sigma}}(P_{\mathcal{K}_\sigma} \pi_\sigma(A) P_{\mathcal{K}_\sigma}) = \omega_\infty(A) \mathbb{I}_{\mathcal{K}_\sigma}, \quad A \in \mathcal{A}_\sigma.$$

Furthermore, by the positivity and the unitarity of $T_{\mathbf{v}_\sigma}$, there exists a $T_{\mathbf{v}_\sigma}$ -invariant state ρ_σ on $M_{m_\sigma} \simeq B(\mathcal{K}_\sigma)$ such that

$$P_{\{1\}}^{T_{\mathbf{v}_\sigma}}(\cdot) = \rho_\sigma(\cdot) \mathbb{I}_{\mathcal{K}_\sigma}.$$

\square

Next we consider the restriction $(\mathbf{v}_\sigma)_{s(\rho_\sigma)}$. (Recall the definitions in Subsection 1.2 of Part I.)

Lemma 3.3. *Assume [A1], [A3], and [A4]. For $\sigma = L, R$, let $\mathbf{v}_\sigma = (v_{1,\sigma}, \dots, v_{n,\sigma})$ be the matrices given in Notation 2.5. Then for ρ_σ given in Lemma 3.2, we have the followings.*

1.

$$s(\rho_\sigma)v_{\mu\sigma} = s(\rho_\sigma)v_{\mu\sigma}s(\rho_\sigma), \quad \mu = 1, \dots, n. \quad (8)$$

2. $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}$ is a unital primitive CP map on $B(s(\rho_\sigma)\mathcal{K}_\sigma)$.

3. Define a linear map $\mathbb{E}^{(\sigma)} : M_n \otimes B(s(\rho_\sigma)\mathcal{K}_\sigma) \rightarrow B(s(\rho_\sigma)\mathcal{K}_\sigma)$ by

$$\mathbb{E}^{(\sigma)} \left(e_{\mu\nu}^{(n)} \otimes X \right) := (v_{\mu\sigma})_{s(\rho_\sigma)} X \left((v_{\nu\sigma})_{s(\rho_\sigma)} \right)^*, \quad X \in B(s(\rho_\sigma)\mathcal{K}_\sigma).$$

Then $(B(s(\rho_\sigma)\mathcal{K}_\sigma), \mathbb{E}^{(\sigma)}, \rho_\sigma|_{B(s(\rho_\sigma)\mathcal{K}_\sigma)})$ is a minimal standard triple σ -generating ω_∞ .

(See Appendix C for the definitions.)

Proof. 1 : As ρ_σ is a $T_{\mathbf{v}_\sigma}$ -invariant state, we have $\sum_{\mu=1}^n v_{\mu\sigma}^* \tilde{\rho}_\sigma v_{\mu\sigma} = \tilde{\rho}_\sigma$, for the density matrix $\tilde{\rho}_\sigma$ of ρ_σ . This implies (8).

2. : The map $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}$ is clearly CP on $B(s(\rho_\sigma)\mathcal{K}_\sigma)$ but it is also unital because

$$T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}(s(\rho_\sigma)) = s(\rho_\sigma)T_{\mathbf{v}_\sigma}(s(\rho_\sigma))s(\rho_\sigma) = s(\rho_\sigma)T_{\mathbf{v}_\sigma}(\mathbb{I}_{\mathcal{K}_\sigma})s(\rho_\sigma) = s(\rho_\sigma).$$

Here we used (8) for the second equality. For any $\lambda \in \sigma \left(T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}} \right) \cap \mathbb{T}$, there exists a nonzero $X \in B(s(\rho_\sigma)\mathcal{K}_\sigma)$ such that $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}(X) = \lambda X$. Using (8) and Lemma 3.2 again, we obtain

$$\lambda^N X = T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}^N(X) = s(\rho_\sigma)T_{\mathbf{v}_\sigma}^N(X)s(\rho_\sigma) \rightarrow \rho_\sigma(X)s(\rho_\sigma), \quad N \rightarrow \infty.$$

From this, we conclude $\lambda = 1$ and $X \in \mathbb{C}s(\rho_\sigma)$. In other words, we have $\sigma \left(T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}} \right) \cap \mathbb{T} = \{1\}$

and $P_{\{1\}}^{T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}} (B(s(\rho_\sigma)\mathcal{K}_\sigma)) = \mathbb{C}s(\rho_\sigma)$. The restriction $\rho_\sigma|_{B(s(\rho_\sigma)\mathcal{K}_\sigma)}$ is $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}$ -invariant faithful state on $B(s(\rho_\sigma)\mathcal{K}_\sigma)$. Therefore, $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}$ is primitive from Lemma C.4 of [BO].

3 : It is clear from 2. that $(B(s(\rho_\sigma)\mathcal{K}_\sigma), \mathbb{E}^{(\sigma)}, \rho_\sigma|_{B(s(\rho_\sigma)\mathcal{K}_\sigma)})$ is a standard triple. It is minimal because $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}}$ is primitive. For any $a_1, a_2 \in \mathbb{Z}$ with $a_1 \leq a_2$ and $A \in \mathcal{A}_{[a_1, a_2]}$, choose $l \in \mathbb{N}$ so that $\tau_l^{(\sigma)}(A) \in \mathcal{A}_\sigma$. We have by (8) and Lemma 2.4, 3.2

$$\begin{aligned} \omega_\infty(A)\mathbb{I}_{\mathcal{K}_\sigma} &= \omega_\infty \left(\tau_l^{(\sigma)}(A) \right) \mathbb{I}_{\mathcal{K}_\sigma} = P_{\{1\}}^{T_{\mathbf{v}_\sigma}} \left(P_{\mathcal{K}_\sigma} \pi_\sigma \left(\tau_l^{(\sigma)}(A) \right) P_{\mathcal{K}_\sigma} \right) \\ &= \sum_{\mu^{(a_2-a_1+1)}, \nu^{(a_2-a_1+1)}} \left\langle \widehat{\psi}_{\mu^{(a_2-a_1+1)}, \tau_{-a_1}}(A) \widehat{\psi}_{\nu^{(a_2-a_1+1)}} \right\rangle \rho_\sigma \left(\widehat{v}_{s(\rho_\sigma), \mu^{(a_2-a_1+1), \sigma}} \left(\widehat{v}_{s(\rho_\sigma), \nu^{(a_2-a_1+1), \sigma}} \right)^* \right) \mathbb{I}_{\mathcal{K}_\sigma}. \end{aligned}$$

This means that $(B(s(\rho_\sigma)\mathcal{K}_\sigma), \mathbb{E}^{(\sigma)}, \rho_\sigma|_{B(s(\rho_\sigma)\mathcal{K}_\sigma)})$ σ -generates ω_∞ . \square

Lemma 3.4. Let \mathcal{K} be a finite dimensional Hilbert space, $n \in \mathbb{N}$, and $\{v_i\}_{i=1}^n$ a set of elements in $B(\mathcal{K})$. We say a subspace W of \mathcal{K} is $\{v_i^*\}_{i=1}^n$ -invariant if $v_i^*W \subset W$ for all $i = 1, \dots, n$. Let \mathcal{K}'_0 be a $\{v_i^*\}_{i=1}^n$ -invariant subspace of \mathcal{K} . Suppose that \mathcal{K}'_0 does not have any proper nonzero subspace which is $\{v_i^*\}_{i=1}^n$ -invariant. Then there exists $k \in \mathbb{N} \cup \{0\}$ and a finite sequence of subspaces $\{\mathcal{K}_a\}_{a=0}^k$ of \mathcal{K} satisfying the followings:

(i) $\mathcal{K}_0 = \mathcal{K}'_0$.

(ii) $\mathcal{K}_0 \subsetneq \mathcal{K}_1 \subsetneq \dots \subsetneq \mathcal{K}_k = \mathcal{K}$.

(iii) $v_i^* \mathcal{K}_a \subset \mathcal{K}_a$, for any $i = 1, \dots, n$, and $a = 0, \dots, k$.

(iv) For any $a = 0, \dots, k-1$, there is no proper intermediate subspace between \mathcal{K}_a and \mathcal{K}_{a+1} which is $\{v_i^*\}_{i=1}^n$ -invariant. .

Furthermore, set $\mathcal{K}_{-1} := \{0\}$ and let p_a be the orthogonal projection onto \mathcal{K}_a , for $a = -1, \dots, k$. Set $r_a := p_a - p_{a-1}$ and $\omega_a := (\omega_{i,a})_{i=1}^n$, $\omega_{i,a} = r_a v_i r_a$ for all $a = 0, \dots, k$. Then for each $a = 0, \dots, k$, T_{ω_a} is an irreducible CP map on $B(r_a \mathcal{K})$ and

$$\omega_{i_1 a} \omega_{i_2 a} \cdots \omega_{i_l a} = p_a v_{i_1} v_{i_2} \cdots v_{i_l} \overline{p_{a-1}}, \quad l \in \mathbb{N}, \quad i_1, \dots, i_l \in \{1, \dots, n\}.$$

Remark 3.5. Here, a proper intermediate space between W_1 and W_2 means a space W such that $W_1 \subsetneq W \subsetneq W_2$.

Proof. We consider the following proposition for $b \in \mathbb{N} \cup \{0\}$:

(P_b): There exists a finite sequence of subspaces $\{\mathcal{K}_a\}_{a=0}^b$ of \mathcal{K} satisfying the followings :

- (i) $\mathcal{K}_0 = \mathcal{K}'_0$.
- (ii) $\mathcal{K}_0 \subsetneq \mathcal{K}_1 \subsetneq \cdots \subsetneq \mathcal{K}_b$.
- (iii) $v_i^* \mathcal{K}_a \subset \mathcal{K}_a$, for any $i = 1, \dots, n$, and $a = 0, \dots, b$.
- (iv) For any $0 \leq a \leq b-1$, there is no proper intermediate subspace between \mathcal{K}_a and \mathcal{K}_{a+1} which is $\{v_i^*\}_{i=1}^n$ -invariant. .

Clearly (P_0) holds. Suppose that (P_b) holds and $\mathcal{K}_b \neq \mathcal{K}$. We claim (P_{b+1}) holds. If there is no intermediate subspace which is $\{v_i^*\}_{i=1}^n$ -invariant, between \mathcal{K}_b and \mathcal{K} , then $\mathcal{K}_{b+1} := \mathcal{K}$ satisfies the condition (P_{b+1}). If there is a proper intermediate subspace \mathcal{H}_1 which is $\{v_i^*\}_{i=1}^n$ -invariant, between \mathcal{K}_b and \mathcal{K} , then we have $\dim \mathcal{K}_b < \dim \mathcal{H}_1 < \dim \mathcal{K}$ and $\mathcal{K}_b \subsetneq \mathcal{H}_1 \subsetneq \mathcal{K}$. Set $\mathcal{K}_{b+1} := \mathcal{H}_1$ if there is no proper intermediate subspace which is $\{v_i^*\}_{i=1}^n$ -invariant, between \mathcal{K}_b and \mathcal{H}_1 , and (P_{b+1}) holds. Otherwise, there exists a proper intermediate subspace \mathcal{H}_2 which is $\{v_i^*\}_{i=1}^n$ -invariant, between \mathcal{K}_b and \mathcal{H}_1 , and $\dim \mathcal{K}_b < \dim \mathcal{H}_2 < \dim \mathcal{H}_1 < \dim \mathcal{K}$. This procedure terminates at some point because $\dim \mathcal{K}$ is finite and at each step, the dimension of \mathcal{H}_i decreases at least 1. Suppose that this iteration terminates at l -th step, and we obtain a subspace \mathcal{H}_l . Setting $\mathcal{K}_{b+1} = \mathcal{H}_l$, we obtain (P_{b+1}). Note that if (P_b) holds for some $\{\mathcal{K}_a\}_{a=0}^b$, then we have $b \leq \dim \mathcal{K}_b \leq \dim \mathcal{K}$. Therefore, there exists $k \in \mathbb{N} \cup \{0\}$ and subspaces $\{\mathcal{K}_a\}_{a=0}^k$ satisfying (P_k) and $\mathcal{K}_k = \mathcal{K}$. This proves the first part of the Lemma.

In order to show that T_{ω_a} is an irreducible CP map on $B(r_a \mathcal{K})$ for $a = 0, \dots, k$, it suffices to show that if a projection p in $B(r_a \mathcal{K})$ satisfies $T_{\omega_a}(pB(r_a \mathcal{K})p) \subset pB(r_a \mathcal{K})p$, then $p = r_a$ or $p = 0$. See Lemma C.2 [O]. To do so, we assume that there exists a projection p in $B(r_a \mathcal{K})$ such that $T_{\omega_a}(pB(r_a \mathcal{K})p) \subset pB(r_a \mathcal{K})p$ and $p \neq 0, r_a$, and show a contradiction. By $T_{\omega_a}(pB(r_a \mathcal{K})p) \subset pB(r_a \mathcal{K})p$, we have

$$0 = (r_a - p)T_{\omega_a}(p)(r_a - p) = \sum_{i=1}^n (r_a - p)w_{ia}p\omega_{ia}^*(r_a - p).$$

Hence we obtain

$$pv_i^*(r_a - p) = p\omega_{ia}^*(r_a - p) = 0, \quad i = 1, \dots, n. \quad (9)$$

Set $\mathcal{K}' := \mathcal{K}_{a-1} + (r_a - p)\mathcal{K}_a$. Note that \mathcal{K}_{a-1} and $(r_a - p)\mathcal{K}_a$ are orthogonal to each other and \mathcal{K}' is a subspace such that $\mathcal{K}_{a-1} \subsetneq \mathcal{K}' \subsetneq \mathcal{K}_a$. We claim that \mathcal{K}' is $\{v_i^*\}_{i=1}^n$ -invariant: for any $\xi \in \mathcal{K}'$, we have an orthogonal decomposition $\xi = p_{a-1}\xi + (r_a - p)\xi$. For the first term, we have $v_i^*p_{a-1}\xi \in v_i^*\mathcal{K}_{a-1} \subset \mathcal{K}_{a-1} \subset \mathcal{K}'$ by the v_i^* -invariance of \mathcal{K}_{a-1} . For the second term, we have

$$v_i^*(r_a - p)\xi = p_a v_i^*(r_a - p)\xi = p_{a-1} v_i^*(r_a - p)\xi + (r_a - p)v_i^*(r_a - p)\xi + p v_i^*(r_a - p)\xi,$$

by the v_i^* -invariance of \mathcal{K}_a . The first term on the right hand side is clearly in $\mathcal{K}_{a-1} \subset \mathcal{K}'$ and the second term is in $(r_a - p)\mathcal{K}_a \subset \mathcal{K}'$. The last term is 0 because of (9). Hence we prove the claim. This means that \mathcal{K}' is a proper intermediate subspace which is $\{v_i^*\}_{i=1}^n$ -invariant, between \mathcal{K}_a and \mathcal{K}_{a+1} . This is the contradiction.

To show the last equality, note that the $\{v_i^*\}$ -invariance of \mathcal{K}_a implies that $p_a v_i = p_a v_i p_a$. From the analogous property of p_{a-1} , we also have $\overline{p_{a-1} v_i p_{a-1}} = v_i \overline{p_{a-1}}$. From this we obtain the last equality. \square

Notation 3.6. We assume [A1],[A3], and [A4]. We denote the density matrix of ρ_σ by $\tilde{\rho}_\sigma$. Set $n_0^{(\sigma)} = \text{rank } s(\rho_\sigma)$ for $\sigma = R, L$. Let us consider \mathcal{K}_σ , $v_{i\sigma}$ and $s(\rho_\sigma)\mathcal{K}_\sigma$, given in Lemma 2.4, Notation 2.5 and Lemma 3.2 for $\sigma = L, R$. We set $\tilde{v}_{\mu\sigma} := s(\rho_\sigma)v_{\mu\sigma}s(\rho_\sigma)$, for $\mu = 1, \dots, n$. We would like to apply Lemma 3.4 to $\mathcal{K} = \mathcal{K}_\sigma$, $v_i = v_{i\sigma}$ and $\mathcal{K}'_0 = s(\rho_\sigma)\mathcal{K}_\sigma$. Note that by (8), $\mathcal{K}'_0 = s(\rho_\sigma)\mathcal{K}_\sigma$ is $\{v_\mu^*\}_{\mu=1}^n$ -invariant. Furthermore, as $T_{(v_\sigma)s(\rho_\sigma)}$ is primitive (Lemma 3.3, 2), for $l \in \mathbb{N}$ large enough, we have $\mathcal{K}_l((v_\sigma)s(\rho_\sigma)) = B(s(\rho_\sigma)\mathcal{K}_\sigma)$. From this, $\mathcal{K}'_0 = s(\rho_\sigma)\mathcal{K}_\sigma$ does not have any nonzero proper subspace which is $\{v_\mu^*\}_{\mu=1}^n$ -invariant. Therefore, we may apply Lemma 3.4. We then obtain $k_\sigma \in \mathbb{N} \cup \{0\}$ and subspaces $\mathcal{K}_{a,\sigma}$, $a = 0, \dots, k_\sigma$ satisfying the properties (i)-(iv) given in Lemma 3.4. Furthermore, set $\mathcal{K}_{-1\sigma} := \{0\}$ and let $p_{a\sigma}$ be the orthogonal projection onto $\mathcal{K}_{a\sigma}$ on \mathcal{K}_σ , for $a = -1, \dots, k_\sigma$. Set $r_{a\sigma} := p_{a\sigma} - p_{a-1,\sigma}$ for $a = 0, \dots, k_\sigma$ and $\omega_{a,\sigma} := (\omega_{i,a,\sigma})_i$, $\omega_{i,a,\sigma} = r_{a\sigma} v_{i\sigma} r_{a\sigma}$, $i = 1, \dots, n$, $a = 0, \dots, k_\sigma$. Then for each $a = 0, \dots, k_\sigma$, $T_{\omega_{a\sigma}}$ is an irreducible CP map on $B(r_{a\sigma}\mathcal{K}_\sigma)$ and

$$\omega_{i_1 a \sigma} \omega_{i_2 a \sigma} \cdots \omega_{i_l a \sigma} = p_{a\sigma} v_{i_1 \sigma} v_{i_2 \sigma} \cdots v_{i_l \sigma} \overline{p_{a-1 \sigma}}, \quad l \in \mathbb{N}, \quad i_1, \dots, i_l \in \{1, \dots, n\}. \quad (10)$$

In particular, either $T_{\omega_{a,\sigma}} = 0$ and $r_{a\sigma}$ is rank one, or $T_{\omega_{a,\sigma}}$ is a nonzero irreducible operator. For the latter case, $r_{T_{\omega_{a,\sigma}}} > 0$ and there exists a strictly positive element $t_{a\sigma}$ in $B(r_{a\sigma}\mathcal{K}_\sigma)$ such that $T_{\omega_{a,\sigma}}(t_{a\sigma}) = r_{T_{\omega_{a,\sigma}}} \cdot t_{a\sigma}$ and any eigenvector of $T_{\omega_{a,\sigma}}$ corresponding to $r_{T_{\omega_{a,\sigma}}}$ belongs to $\mathbb{C}t_{a\sigma}$. We define $\mathbf{u}_{a\sigma} = (u_{\mu a \sigma})_\mu$ by

$$u_{\mu a \sigma} := r_{T_{\omega_{a,\sigma}}}^{-\frac{1}{2}} t_{a\sigma}^{-\frac{1}{2}} \omega_{\mu a \sigma} t_{a\sigma}^{\frac{1}{2}}, \quad \mu = 1, \dots, n.$$

By this definition, $T_{\mathbf{u}_{a\sigma}}$ is a unital irreducible CP map. Therefore, applying Lemma D.1, we have

1. There exists a $b_{a\sigma} \in \mathbb{N}$ such that $\sigma(T_{\mathbf{u}_{a\sigma}}) \cap \mathbb{T} = \left\{ \exp\left(\frac{2\pi i}{b_{a\sigma}} k\right) \mid k = 0, \dots, b_{a\sigma} - 1 \right\}$.
2. For any $\lambda \in \sigma(T_{\mathbf{u}_{a\sigma}}) \cap \mathbb{T}$, λ is a nondegenerate eigenvalue of $T_{\mathbf{u}_{a\sigma}}$.
3. There exists a unitary matrix $U_{a\sigma} \in B(r_{a\sigma}\mathcal{K}_\sigma)$ such that

$$T_{\mathbf{u}_{a\sigma}}(U_{a\sigma}^k) = \exp\left(\frac{2\pi i}{b_{a\sigma}} k\right) U_{a\sigma}^k, \quad k = 0, \dots, b_{a\sigma} - 1.$$

4. The unitary matrix $U_{a\sigma}$ in 3 has a spectral decomposition

$$U_{a\sigma} = \sum_{k=0}^{b_{a\sigma}-1} \exp\left(\frac{2\pi i}{b_{a\sigma}} k\right) Q_k,$$

with spectral projections satisfying

$$T_{\mathbf{u}_{a\sigma}}(Q_k) = Q_{k-1}, \quad k \mod b_{a\sigma}.$$

5. The restriction $T_{\mathbf{u}_{a\sigma}}|_{Q_k B(r_{a\sigma}\mathcal{K}_\sigma) Q_k}$ of $T_{\mathbf{u}_{a\sigma}}$ on $Q_k B(r_{a\sigma}\mathcal{K}_\sigma) Q_k$ defines a primitive unital CP map on $Q_k B(r_{a\sigma}\mathcal{K}_\sigma) Q_k$.

6. There exists a faithful $T_{\mathbf{u}_{a\sigma}}$ -invariant state $\varphi_{a\sigma}$.

Lemma 3.7. *For any $a \geq 1$, we have $r_{T_{\omega_{a,\sigma}}} < 1$.*

Proof. Using (10), we have for $a \geq 1$,

$$\begin{aligned} r_{T_{\omega_{a,\sigma}}} &= \lim_{N \rightarrow \infty} \|T_{\omega_{a\sigma}}^N(r_{a\sigma})\|^{\frac{1}{N}} = \lim_{N \rightarrow \infty} \|r_{a\sigma} T_{\mathbf{v}_\sigma}^N(r_{a\sigma}) r_{a\sigma}\|^{\frac{1}{N}} \leq \limsup_{N \rightarrow \infty} \left(\|\rho_\sigma(r_{a\sigma}) r_{a\sigma}\| + \|T_{\mathbf{v}_\sigma}^N(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}})\| \right)^{\frac{1}{N}} \\ &= \limsup_{N \rightarrow \infty} \left(\|T_{\mathbf{v}_\sigma}^N(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}})\| \right)^{\frac{1}{N}} < 1. \end{aligned}$$

□

As a result of [A5], we eliminate the possibility $T_{\omega_{a,\sigma}} = 0$.

Lemma 3.8. *Assume [A1],[A3],[A4], and [A5], and consider the setting in Notation 3.6. Then for any $a = 1, \dots, k_\sigma$, we have $T_{\omega_{a,\sigma}} \neq 0$.*

Proof. If the claim does not hold, there exists an $a_0 = 1, \dots, k_\sigma$ such that $T_{\omega_{a_0,\sigma}} = 0$. As stated in Notation 3.6, in this case, $r_{a_0\sigma}$ is rank one. Let ξ be a unit vector in the one dimensional space $r_{a_0\sigma}\mathcal{K}_\sigma$. Then $\psi = \langle \xi, \pi_\sigma(\cdot)\xi \rangle$ is an element of $\mathcal{S}_\sigma(H)$ by Lemma 2.4.

Because of the choice of a_0 , we know that $\overline{p_{a_0-1,\sigma} v_{\mu^{(l)},\sigma}^*} p_{a_0,\sigma} = 0$, using (10). Therefore, we have $v_{\mu^{(l)},\sigma}^* \mathcal{K}_{a_0,\sigma} \subset \mathcal{K}_{a_0-1,\sigma}$, for any $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$, and $l \in \mathbb{N}$. In particular, ξ and $v_{\mu^{(l)},\sigma}^* \xi$ are orthogonal, for any $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ and $l \in \mathbb{N}$. From this, we obtain

$$\psi \circ \tau_l^{(\sigma)} = \sum_{\mu^{(l)}} \langle \xi, \widehat{v_{\mu^{(l)},\sigma} \pi_\sigma(\cdot)} \widehat{v_{\mu^{(l)},\sigma}^* \xi} \rangle = \sum_{\mu^{(l)}} \langle \widehat{v_{\mu^{(l)},\sigma}^* \xi}, \pi_\sigma(\cdot) \widehat{v_{\mu^{(l)},\sigma}^* \xi} \rangle.$$

Let p be the orthogonal projection onto $\mathbb{C}\xi$. As ξ and $v_{\mu^{(l)},\sigma}^* \xi$ are orthogonal, we have $p v_{\mu^{(l)},\sigma}^* \xi = 0$. Since φ_σ is pure, we have $\pi_\sigma(\mathcal{A}_\sigma)'' = B(\mathcal{H}_\sigma)$. Therefore, by Kaplansky's density Theorem, there exists a net $\{x_\alpha\}$ in the unit ball of $\mathcal{A}_{\sigma+}$ such that $\sigma w - \lim_\alpha \pi_\sigma(x_\alpha) = p$. For this net, we have $\lim_\alpha \psi \circ \tau_l^{(\sigma)}(x_\alpha) = 0$, and $\lim_\alpha \psi(x_\alpha) = 1$, for any $l \in \mathbb{N}$. Note also that $-1 \leq 2x_\alpha - 1 \leq 1$, hence $\|2x_\alpha - 1\| \leq 1$.

This implies

$$\left\| \psi - \psi \circ \tau_l^{(\sigma)} \right\| \geq \lim_\alpha \left| \left(\psi - \psi \circ \tau_l^{(\sigma)} \right) (2x_\alpha - 1) \right| = 2, \quad l \in \mathbb{N}.$$

This contradicts [A5].

□

4 Primitivity of $T_{\mathbf{u}_{a\sigma}}$

In this section, we show that the number $b_{a\sigma}$, given in 3.6 is 1. In other words, $T_{\mathbf{u}_{a\sigma}}$ is primitive. More precisely, we show the following Lemma.

Lemma 4.1. *We assume [A1],[A3],[A4], and [A5] and use Notation 2.5 and Notation 3.6. Then for any $a = 0, \dots, k_\sigma$, there exist a unitary $V_{a\sigma} : \mathbb{C}^{n_0^{(\sigma)}} \rightarrow r_{a\sigma}\mathcal{K}_\sigma$ and $c_{a\sigma} \in \mathbb{T}$ such that*

$$u_{\mu a\sigma} = c_{a\sigma} V_{a\sigma} \widetilde{v_{\mu\sigma}} V_{a\sigma}^*, \quad \mu = 1, \dots, n.$$

In particular, $T_{\mathbf{u}_{a\sigma}}$ is primitive and $\text{rank } r_{a\sigma} = n_0^{(\sigma)}$.

This is because of the uniqueness of the bulk ground state [A3], which implies the following Lemma.

Lemma 4.2. *We assume [A1], [A3], [A4] [A5], and use Notation 2.5, Notation 3.6. Then $(B(r_{a\sigma}\mathcal{K}_\sigma), \mathbf{u}_{a\sigma}, \varphi_{a\sigma})$ σ -generates ω_∞ .*

Proof. We denote $\mathbf{v} = \mathbf{v}_\sigma$, $\boldsymbol{\omega} = \boldsymbol{\omega}_{a\sigma}$, and $\mathbf{u} = \mathbf{u}_{a\sigma}$, in this proof, for simplicity. Define $\mathbb{E}^{(a\sigma)} : M_n \otimes B(r_{a\sigma}\mathcal{K}_\sigma) \rightarrow B(r_{a\sigma}\mathcal{K}_\sigma)$ by

$$\mathbb{E}^{(a\sigma)} \left(e_{\mu\nu}^{(n)} \otimes X \right) := u_\mu X u_\nu^*, \quad X \in B(r_{a\sigma}\mathcal{K}_\sigma). \quad (11)$$

Then $(B(r_{a\sigma}\mathcal{K}_\sigma), \mathbb{E}^{(a\sigma)}, \varphi_{a\sigma})$ is a standard triple, and σ -generates a state $\tilde{\omega}_{a,\sigma}$ on $\mathcal{A}_\mathbb{Z}$. We claim that $\tilde{\omega}_{a,\sigma} = \omega_\infty$. To see this, it suffices to show that $\tilde{\omega}_{a,\sigma}(\tau_x(h)) = 0$ for all $x \in \mathbb{Z}$ because of [A3] and Lemma 2.1, 1.

By the definition, we have for any $x \in \mathbb{Z}$,

$$\begin{aligned} 0 &\leq \tilde{\omega}_{a,\sigma}(\tau_x(h)) = \sum_{\mu^{(m)} \nu^{(m)}} \left\langle \widehat{\psi_{\mu^{(m)}}}, h \widehat{\psi_{\nu^{(m)}}} \right\rangle \varphi_{a\sigma} \left(\widehat{u_{\mu^{(m),\sigma}}} \left(\widehat{u_{\nu^{(m),\sigma}}} \right)^* \right) \\ &= r_{T_\omega}^{-m} \sum_{\mu^{(m)} \nu^{(m)}} \left\langle \widehat{\psi_{\mu^{(m)}}}, h \widehat{\psi_{\nu^{(m)}}} \right\rangle \varphi_{a\sigma} \left(t_{a\sigma}^{-\frac{1}{2}} \widehat{v_{\mu^{(m),\sigma}}} t_{a\sigma} \left(\widehat{v_{\nu^{(m),\sigma}}} \right)^* t_{a\sigma}^{-\frac{1}{2}} \right) \\ &= r_{T_\omega}^{-m} \sum_{\lambda^{(m)}} \varphi_{a\sigma} \left(t_{a\sigma}^{-\frac{1}{2}} \left(\sum_{\mu^{(m)}} \left\langle \widehat{\psi_{\mu^{(m)}}}, h^{\frac{1}{2}} \widehat{\psi_{\lambda^{(m)}}} \right\rangle \widehat{v_{\mu^{(m),\sigma}}} \right) t_{a\sigma} \left(\sum_{\nu^{(m)}} \left\langle \widehat{\psi_{\nu^{(m)}}}, h^{\frac{1}{2}} \widehat{\psi_{\lambda^{(m)}}} \right\rangle \widehat{v_{\nu^{(m),\sigma}}} \right)^* t_{a\sigma}^{-\frac{1}{2}} \right) \\ &\leq \|t_{a\sigma}\| r_{T_\omega}^{-m} \sum_{\lambda^{(m)}} \varphi_{a\sigma} \left(t_{a\sigma}^{-\frac{1}{2}} \left(\sum_{\mu^{(m)}} \left\langle \widehat{\psi_{\mu^{(m)}}}, h^{\frac{1}{2}} \widehat{\psi_{\lambda^{(m)}}} \right\rangle \widehat{v_{\mu^{(m),\sigma}}} \right) \left(\sum_{\nu^{(m)}} \left\langle \widehat{\psi_{\nu^{(m)}}}, h^{\frac{1}{2}} \widehat{\psi_{\lambda^{(m)}}} \right\rangle \widehat{v_{\nu^{(m),\sigma}}} \right)^* t_{a\sigma}^{-\frac{1}{2}} \right) \\ &= \|t_{a\sigma}\| r_{T_\omega}^{-m} \sum_{\mu^{(m)}} \sum_{\nu^{(m)}} \left\langle \widehat{\psi_{\mu^{(m)}}}, h \widehat{\psi_{\nu^{(m)}}} \right\rangle \varphi_{a\sigma} \left(t_{a\sigma}^{-\frac{1}{2}} \left(\widehat{v_{\mu^{(m),\sigma}}} \right) \left(\widehat{v_{\nu^{(m),\sigma}}} \right)^* t_{a\sigma}^{-\frac{1}{2}} \right) \\ &= \|t_{a\sigma}\| r_{T_\omega}^{-m} \varphi_{a\sigma} \left(t_{a\sigma}^{-\frac{1}{2}} P_{\mathcal{K}_\sigma} \pi_\sigma(\tau_y(h)) P_{\mathcal{K}_\sigma} t_{a\sigma}^{-\frac{1}{2}} \right) = 0, \end{aligned}$$

where $y = 0$ if $\sigma = R$ and $y = -m$ if $\sigma = L$. \square

Remark 4.3. Let $\mathfrak{B}_{a\sigma}$ be the minimal C^* -subalgebra of $B(r_{a\sigma}\mathcal{K}_\sigma)$ which contains \mathbb{I} and is \mathbb{E}_A -invariant for any $A \in M_n$. Then for $\mathbb{E}^{(a\sigma)}$ given by (11), $(\mathfrak{B}_{a\sigma}, \mathbb{E}^{(a\sigma)}|_{M_n \otimes \mathfrak{B}_{a\sigma}}, \varphi_{a\sigma}|_{\mathfrak{B}_{a\sigma}})$ is a minimal standard triple σ -generating ω_∞ , from Lemma 4.2. The eigenspace of 1 for $T_{\mathbf{u}_{a\sigma}} = \mathbb{E}_\mathbb{I}^{(a\sigma)}$ is $\mathbb{C}\mathbb{I}$. Recall that $(B(s(\rho_\sigma)\mathcal{K}_\sigma), \mathbb{E}^{(\sigma)}, \rho_\sigma|_{B(s(\rho_\sigma)\mathcal{K}_\sigma)})$ is a minimal standard triple σ -generating ω_∞ . The eigenspace of 1 for $T_{(\mathbf{v}_\sigma)_{s(\rho_\sigma)}} = \mathbb{E}_\mathbb{I}^{(\sigma)}$ is $\mathbb{C}\mathbb{I}$ (Lemma 3.3). For each $N \in \mathbb{N}$, let D_N be the density matrix of $\omega_\infty|_{\mathcal{A}_{[0, N-1]}}$. We have $\sup_N \text{rank } D_N < \infty$ because of [A1] and Lemma 2.1. By Theorem C.3, this implies the existence of a $*$ -isomorphism $\Theta_{a\sigma} : B(s(\rho_\sigma)\mathcal{K}_\sigma) \rightarrow \mathfrak{B}_{a\sigma}$ satisfying

$$\mathbb{E}^{(a\sigma)} \circ (id_{M_n} \otimes \Theta_{a\sigma}) = \Theta_{a\sigma} \circ \mathbb{E}^{(\sigma)}.$$

This condition can be written

$$\Theta_{a\sigma} \left((v_{\mu\sigma})_{s(\rho_\sigma)} X \left((v_{\nu\sigma})_{s(\rho_\sigma)} \right)^* \right) = u_{\mu a\sigma} \Theta_{a\sigma}(X) (u_{\nu a\sigma})^*, \quad \mu, \nu = 1, \dots, n, \quad X \in B(s(\rho_\sigma)\mathcal{K}_\sigma).$$

We apply the following Lemma to this situation.

Lemma 4.4. *Let $n, d_1, d_2 \in \mathbb{N}$. Let $\mathbf{v}^{(i)} = (v_\mu^{(i)})_{\mu=1}^n \in M_{d_i}^{\times n}$, for $i = 1, 2$. Assume that $T_{\mathbf{v}^{(1)}}$ is a primitive unital CP map and that $T_{\mathbf{v}^{(2)}}$ is an irreducible unital CP map. Furthermore, assume that there exists an injective unital $*$ -homomorphism $\Theta : M_{d_1} \rightarrow M_{d_2}$ such that*

$$\Theta \left(v_\mu^{(1)} X \left(v_\nu^{(1)} \right)^* \right) = v_\mu^{(2)} \Theta(X) \left(v_\nu^{(2)} \right)^*, \quad \mu, \nu = 1, \dots, n, \quad X \in M_{d_1}. \quad (12)$$

Then $d_1 = d_2$, and $T_{\mathbf{v}^{(2)}}$ is primitive. There exists a unitary $W : \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_1}$ and a complex number $c \in \mathbb{T}$ such that

$$W v_\mu^{(2)} W^* = c v_\mu^{(1)}, \quad \mu = 1, \dots, n.$$

Proof. Applying Lemma D.1 to $T_{\mathbf{v}^{(2)}}$, we obtain b, U, P_k satisfying 1.-6. of Lemma D.1. Note from 4. of Lemma D.1, we have

$$v_\mu^{(2)} P_k = P_{k-1} v_\mu^{(2)}, \quad k = 0, \dots, b-1 \pmod{b}, \quad \mu = 1, \dots, n. \quad (13)$$

First we claim that

$$\Theta(M_{d_1}) \subset \bigoplus_{k=0}^{b-1} P_k M_{d_2} P_k. \quad (14)$$

To see this, recall that $T_{\mathbf{v}^{(1)}}$ is primitive. Therefore, there exists an $l_0 \in \mathbb{N}$ such that $\mathcal{K}_{lb}(\mathbf{v}^{(1)}) = M_{d_1}$, for all $l \geq l_0$. From this and (12), (13), we have

$$\begin{aligned} \Theta(M_{d_1}) &= \Theta \left(\mathcal{K}_{lb}(\mathbf{v}^{(1)}) \left(\mathcal{K}_{lb}(\mathbf{v}^{(1)}) \right)^* \right) = \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right) \Theta(1) \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right)^* \\ &= \sum_{k=0}^{b-1} \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right) P_k \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right)^* = \sum_{k=0}^{b-1} P_k \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right) \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right)^* P_k, \end{aligned} \quad (15)$$

for any $l \geq l_0$, proving the claim.

Next we claim that for each $k = 0, \dots, b-1$, there exists a unitary $V_k : \mathbb{C}^{d_1} \rightarrow P_k \mathbb{C}^{d_2}$ such that

$$\Theta(X) P_k = V_k X V_k^*, \quad X \in M_{d_1}.$$

To see this, first note that $\Theta_k : M_{d_1} \rightarrow P_k M_{d_2} P_k$ given by $\Theta_k(X) = \Theta(X) P_k$, $k = 0, \dots, b$ is a $*$ -homomorphism because of the first observation.

The map $T_{\mathbf{v}^{(2)}}^b|_{P_k M_{d_2} P_k}$ is primitive by 5. of Lemma D.1. This fact, combined with (13), implies the existence of an $l_1 \in \mathbb{N}$ such that $P_k \mathcal{K}_{lb}(\mathbf{v}^{(2)}) = P_k M_{d_2} P_k$ for all $l \geq l_1$. Therefore, using (15) and (14), we have for $l \geq \max\{l_1, l_0\}$,

$$P_k M_{d_2} P_k = (P_k M_{d_2} P_k) (P_k M_{d_2} P_k)^* = P_k \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right) \left(\mathcal{K}_{lb}(\mathbf{v}^{(2)}) \right)^* P_k = P_k \Theta(M_{d_1}) P_k = \Theta_k(M_{d_1}).$$

Therefore, Θ_k is a $*$ -homomorphism from M_{d_1} onto $P_k M_{d_2} P_k$. As M_{d_1} is simple, it is also injective. Hence, Θ_k is a $*$ -isomorphism between M_{d_1} and $P_k M_{d_2} P_k$. By Wigner's Theorem, this implies the existence of V_k as we claimed.

Define a linear map $W : \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_1} \otimes \mathbb{C}^b$ by

$$W\xi := \sum_{k=0}^{b-1} V_k^* P_k \xi \otimes \chi_{k+1}^{(b)}, \quad \xi \in \mathbb{C}^{d_2}.$$

It is easy to check that W is unitary and

$$\left(W v_\mu^{(2)} W^* \right) (X \otimes \mathbb{I}) \left(W v_\nu^{(2)} W^* \right)^* = v_\mu^{(1)} X v_\nu^{(1)*} \otimes \mathbb{I}, \quad X \in M_{d_1}, \quad \mu, \nu = 1, \dots, n, \quad (16)$$

from (12). Substituting $X = 1$ and $\mu = \nu$ in (16), we obtain

$$\left(W v_\mu^{(2)} W^*\right) \left(W v_\mu^{(2)} W^*\right)^* = v_\mu^{(1)} v_\mu^{(1)*} \otimes \mathbb{I}, \quad \mu = 1, \dots, n.$$

By the polar decomposition, This means that there exist unitary operators \mathcal{W}_μ , $\mu = 1, \dots, n$ in $M_{d_1} \otimes M_b$ such that

$$W v_\mu^{(2)} W^* = \left(v_\mu^{(1)} \otimes \mathbb{I}\right) \mathcal{W}_\mu. \quad (17)$$

We claim for each $\mu = 1, \dots, n$ that \mathcal{W}_μ has a decomposition

$$\mathcal{W}_\mu = w_\mu^{(1)} + w_\mu^{(2)}, \quad (18)$$

where $w_\mu^{(1)}$ is a unitary in $\left(s_r(v_\mu^{(1)}) M_{d_1} s_r(v_\mu^{(1)})\right) \otimes M_b$ and $w_\mu^{(2)}$ is a unitary in $\left(\overline{s_r(v_\mu^{(1)})} M_{d_1} \overline{s_r(v_\mu^{(1)})}\right) \otimes M_b$. We substitute $X = \overline{s_r(v_\mu^{(1)})}$ in (16) with $\mu = \nu$, and obtain

$$0 = v_\mu^{(1)} \overline{s_r(v_\mu^{(1)})} v_\mu^{(1)*} \otimes \mathbb{I} = \left(W v_\mu^{(2)} W^*\right) \left(\overline{s_r(v_\mu^{(1)})} \otimes \mathbb{I}\right) \left(W v_\mu^{(2)} W^*\right)^* = \left(v_\mu^{(1)} \otimes \mathbb{I}\right) \mathcal{W}_\mu \left(\overline{s_r(v_\mu^{(1)})} \otimes \mathbb{I}\right) \mathcal{W}_\mu^* \left(v_\mu^{(1)} \otimes \mathbb{I}\right)^*.$$

This means $\left(s_r(v_\mu^{(1)}) \otimes \mathbb{I}\right) \mathcal{W}_\mu \left(\overline{s_r(v_\mu^{(1)})} \otimes \mathbb{I}\right) = 0$ which implies the claim.

Assume that $v_\mu^{(1)} \neq 0$. By (16), (17), and (18), we have

$$\left(v_\mu^{(1)} \otimes \mathbb{I}\right) w_\mu^{(1)} (X \otimes \mathbb{I}) w_\mu^{(1)*} \left(v_\mu^{(1)} \otimes \mathbb{I}\right)^* = v_\mu^{(1)} X v_\mu^{(1)*} \otimes \mathbb{I}, \quad X \in M_{d_1}.$$

From this we get

$$w_\mu^{(1)} (X \otimes \mathbb{I}) w_\mu^{(1)*} = X \otimes \mathbb{I}, \quad X \in s_r(v_\mu^{(1)}) M_{d_1} s_r(v_\mu^{(1)}).$$

This means $w_\mu^{(1)} \in s_r(v_\mu^{(1)}) \otimes M_b$. Therefore, there exists a unitary \tilde{w}_μ such that

$$w_\mu^{(1)} = s_r(v_\mu^{(1)}) \otimes \tilde{w}_\mu.$$

We have

$$v_\mu^{(1)} \otimes \tilde{w}_\mu = \left(v_\mu^{(1)} \otimes \mathbb{I}\right) \left(s_r(v_\mu^{(1)}) \otimes \tilde{w}_\mu\right) = \left(v_\mu^{(1)} \otimes \mathbb{I}\right) \mathcal{W}_\mu = W v_\mu^{(2)} W^*, \quad (19)$$

for all $\mu = 1, \dots, n$ with $v_\mu^{(1)} \neq 0$.

The unitary matrices \tilde{w}_μ can be taken independent of μ , $v_\mu^{(1)} \neq 0$. To see this, substitute (19) to (16) and obtain

$$v_\mu^{(1)} X v_\nu^{(1)*} \otimes \tilde{w}_\mu \tilde{w}_\nu^* = v_\mu^{(1)} X v_\nu^{(1)*} \otimes \mathbb{I}, \quad X \in M_{d_1}, \quad \mu, \nu = 1, \dots, n, \quad v_\mu^{(1)} \neq 0, \quad v_\nu^{(1)} \neq 0.$$

If $v_\mu^{(1)}$ and $v_\nu^{(1)}$ are not zero, this equality means that $\tilde{w}_\mu \tilde{w}_\nu^* = 1$, i.e., $\tilde{w}_\mu = \tilde{w}_\nu$. We deote this common \tilde{w}_μ by w . (Note that there exists at least one $v_\mu^{(1)} \neq 0$ because $T_{\mathbf{v}(1)}$ is unital.) Hence we obtain

$$W v_\mu^{(2)} W^* = v_\mu^{(1)} \otimes w, \quad \mu = 1, \dots, n. \quad (20)$$

Note that this also holds for μ with $v_\mu^{(1)} = 0$ because of (17).

Now we prove $b = 1$, i.e., $d_1 = d_2$. Assume that $b \neq 1$. Then because w is unitary, there exists an $x \in \mathbb{M}_b$ such that $w x w^* = x$ and $x \notin \mathbb{C} \mathbb{I}_b$. Set $X := W^* (\mathbb{I} \otimes x) W \in \mathbb{M}_{d_2}$. We have $X \notin \mathbb{C} \mathbb{I}_{d_2}$ and

$$T_{\mathbf{v}^{(2)}}(X) = \sum_{\mu=1}^n W^* W v_{\mu}^{(2)} W^* (\mathbb{I} \otimes x) W v_{\mu}^{(2)*} W^* W = \sum_{\mu=1}^n W^* \left(v_{\mu}^{(1)} v_{\mu}^{(1)*} \otimes w x w^* \right) W = W^* (\mathbb{I} \otimes x) W = X.$$

By 2. of Lemma D.1, 1 is a nondegenerate eigenvalue of $T_{\mathbf{v}^{(2)}}$. This is a contradiction. Therefore, we conclude $b = 1$ and $d_1 = d_2$. In this case, (20) implies the existence of unitary $W : \mathbb{C}^{d_2} \rightarrow \mathbb{C}^{d_1}$ and $c \in \mathbb{T}$ satisfying

$$W v_{\mu}^{(2)} W^* = c v_{\mu}^{(1)} \quad \mu = 1, \dots, n.$$

Clearly, this implies the primitivity of $T_{\mathbf{v}^{(2)}}$. \square

Proof of Lemma 4.1. Recall Remark 4.3. Applying Lemma 4.4 to $\Theta_{a\sigma}$, we obtain a unitary $W_{a\sigma} : r_{a\sigma} \mathcal{K}_{\sigma} \rightarrow s(\rho_{\sigma}) \mathcal{K}_{\sigma}$, and $c_{a\sigma} \in \mathbb{T}$ such that $W_{a\sigma} u_{\mu a\sigma} W_{a\sigma}^* = c_{a\sigma} \widehat{v_{\mu\sigma}}$ for $\mu = 1, \dots, n$. Set $V_{a\sigma} := W_{a\sigma}^*$ and under the identification $\mathbb{C}^{n_0^{(\sigma)}} \simeq s(\rho_{\sigma}) \mathcal{K}_{\sigma}$, we complete the proof. \square

5 The bijectivity of $\Gamma_{l,\mathbf{v}}^{(\sigma)} \Big|_{B(\mathcal{K}_{\sigma})s(\rho_{\sigma})}$

In this section, we prove the following Lemma.

Lemma 5.1. Assume [A1], [A3], [A4], and [A5]. Let $\sigma = L, R$. Let \mathbf{v}_{σ} be the n -tuple of elements in $B(\mathcal{K}_{\sigma})$ given in Notation 2.5 and ρ_{σ} the state given in Lemma 3.2. Then there exists an $l'_{\sigma} \in \mathbb{N}$ such that

$$\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)} \Big|_{B(\mathcal{K}_{\sigma})s(\rho_{\sigma})} : B(\mathcal{K}_{\sigma})s(\rho_{\sigma}) \rightarrow \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma})) = \tau_{y_{\sigma}} \left(s \left(\omega_{\sigma}|_{\mathcal{A}_{\sigma,l}} \right) \right) \bigotimes_{i=0}^{l-1} \mathbb{C}^n$$

is a bijection for any $l \geq l'_{\sigma}$. Here, $y_R = 0$ and $y_L = l$.

We start from the following simple observation.

Lemma 5.2. Assume [A1] and [A4]. Let \mathbf{v}_{σ} be the n -tuple of elements in $B(\mathcal{K}_{\sigma})$ given in Notation 2.5. For $l \in \mathbb{N}$, a unit vector $\xi \in \mathcal{K}_{\sigma}$ and a projection p in $B(\mathcal{K}_{\sigma})$, define $X_{l,\xi,p}^{(\sigma)} \in \mathcal{A}_{[0,l-1]}$ by

$$X_{l,\xi,p}^{(\sigma)} := \sum_{\mu^{(l)}, \nu^{(l)}} \langle \xi, \widehat{v_{\mu^{(l),\sigma}}} p \widehat{v_{\nu^{(l),\sigma}}}^* \xi \rangle \left| \widehat{\psi_{\nu^{(l)}}} \right\rangle \left\langle \widehat{\psi_{\mu^{(l)}}} \right|.$$

Let ω_{ξ} be a state given by $\omega_{\xi} = \langle \xi, \pi_{\sigma}(\cdot) \xi \rangle$. Then $X_{l,\xi,p}^{(\sigma)}$ is positive and

$$s \left(X_{l,\xi,p}^{(\sigma)} \right) \leq \tau_{y_{\sigma}} \left(s \left(\omega_{\xi}|_{\mathcal{A}_{\sigma,l}} \right) \right), \quad \omega_{\xi} \circ \tau_{-y_{\sigma}}(A) = \text{Tr} \left(X_{l,\xi,\mathbb{I}}^{(\sigma)} A \right), \quad A \in \mathcal{A}_{[0,l-1]},$$

where $y_R = 0$ and $y_L = l$. Furthermore, for a unit vector $\eta \in \mathcal{K}_{\sigma}$,

$$X_{l,\xi,|\eta\rangle\langle\eta|}^{(\sigma)} = \left| \Gamma_{l\mathbf{v}_{\sigma}}^{(\sigma)}(|\xi\rangle\langle\eta|) \right\rangle \left\langle \Gamma_{l\mathbf{v}_{\sigma}}^{(\sigma)}(|\xi\rangle\langle\eta|) \right|.$$

Proof. For any $\zeta \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$, we have

$$\langle \zeta, X_{l,\xi,p}^{(\sigma)} \zeta \rangle = \left\| p \left(\sum_{\nu^{(l)}} \langle \widehat{\psi_{\nu^{(l)}}}, \zeta \rangle \widehat{v_{\nu^{(l),\sigma},\sigma}} \right)^* \xi \right\|^2 \geq 0,$$

and

$$\omega_{\xi}(\tau_{-y_{\sigma}}(|\zeta\rangle\langle\zeta|)) = \left\| \left(\sum_{\nu^{(l)}} \langle \widehat{\psi_{\nu^{(l)}}}, \zeta \rangle \widehat{v_{\nu^{(l),\sigma},\sigma}} \right)^* \xi \right\|^2,$$

where $y_R = 0$ and $y_L = l$. The claim of the Lemma can be checked from these equations. \square

Lemma 5.3. Assume [A1] and [A4]. Let ω_{σ} be the state in [A4] and \mathbf{v}_{σ} the n -tuple of elements in $B(\mathcal{K}_{\sigma})$ given in Notation 2.5. Then for any $l \in \mathbb{N}$, $\tau_{y_{\sigma}}(s(\omega_{\sigma}|_{\mathcal{A}_{\sigma,l}}))$ is equal to the orthogonal projection onto $\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma}))$, where $y_R = 0$ and $y_L = l$.

Proof. Let $\rho_{\omega_{\sigma}}$ be the density matrix given by Lemma 2.4. As it is strictly positive, it can be decomposed as $\rho_{\omega_{\sigma}} = \sum_i \lambda_i |\eta_i\rangle\langle\eta_i|$ with numbers $\lambda_i > 0$ and CONS $\{\eta_i\}_i$ of \mathcal{K}_{σ} . By Lemma 5.2, we get

$$\begin{aligned} \omega_{\sigma} \circ \tau_{-y_{\sigma}}(A) &= \sum_i \lambda_i \omega_{\eta_i} \circ \tau_{-y_{\sigma}}(A) = \sum_{ij} \lambda_i \text{Tr} \left(X_{l,\eta_i,|\eta_j\rangle\langle\eta_j|}^{(\sigma)} A \right) \\ &= \sum_{ij} \lambda_i \left\langle \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(|\eta_i\rangle\langle\eta_j|), A \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(|\eta_i\rangle\langle\eta_j|) \right\rangle, \quad A \in \mathcal{A}_{[0,l-1]}, \quad l \in \mathbb{N}. \end{aligned}$$

As $\{\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(|\eta_i\rangle\langle\eta_j|)\}_{ij}$ spans $\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma}))$, this proves the Lemma. \square

Lemma 5.4. Assume [A1], [A3], [A4], and [A5]. Let \mathbf{v}_{σ} be the n -tuple of elements in $B(\mathcal{K}_{\sigma})$ given in Notation 2.5 and ρ_{σ} the state given in Lemma 3.2. Then there exists an $\tilde{l}_{\sigma} \in \mathbb{N}$ such that

$$\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma})s(\rho_{\sigma})) = \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma})), \quad l \geq \tilde{l}_{\sigma}.$$

Proof. Set

$$C_{\sigma} := \inf \left\{ \sigma \left(D_{\omega_{\sigma}|_{\mathcal{A}_{\sigma,l}}} \right) \setminus \{0\} \mid l \in \mathbb{N} \right\} > 0,$$

and

$$\tilde{l}_{\sigma} := \min \left\{ l \in \mathbb{N} \mid \sup_{l': l \leq l'} \left\{ \left\| T_{\mathbf{v}_{\sigma}}^{l'} \left(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_{\sigma}}} \right) \right\| \right\} < \frac{1}{2} C_{\sigma} \right\} \in \mathbb{N}.$$

(Recall [A4] and Lemma 3.2 to see $C_{\sigma} > 0$ and $\tilde{l}_{\sigma} \in \mathbb{N}$.) We assume that there exists an $l \geq \tilde{l}_{\sigma}$ such that $\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma})s(\rho_{\sigma})) \neq \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma}))$ and show a contradiction.

If $\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma})s(\rho_{\sigma})) \neq \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma}))$, then there exists an $X \in B(\mathcal{K}_{\sigma})$ such that $\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(X)$ and $\Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(B(\mathcal{K}_{\sigma})s(\rho_{\sigma}))$ are orthogonal and $\left\| \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(X) \right\| = 1$.

First we show

$$\left| \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(X) \right\rangle \left\langle \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(X) \right| \leq \tau_{y_{\sigma}} \left(s \left(D_{\omega_{\sigma}|_{\mathcal{A}_{\sigma,l}}} \right) \right), \quad (21)$$

where $y_R = 0$ and $y_L = l$. We may represent $X = \sum_{i=1}^{m_{\sigma}} c_i |\xi_i\rangle\langle\eta_i|$, with $\|\xi_i\| = \|\eta_i\| = 1$. From Lemma 5.2, we have

$$\left| \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(|\xi_i\rangle\langle\eta_i|) \right\rangle \left\langle \Gamma_{l,\mathbf{v}_{\sigma}}^{(\sigma)}(|\xi_i\rangle\langle\eta_i|) \right| = X_{l,\xi_i,|\eta_i\rangle\langle\eta_i|}^{(\sigma)},$$

with notation in Lemma 5.2. By Lemma 5.2, and Lemma 2.3 we have

$$s \left(\left| \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(|\xi_i\rangle \langle \eta_i|) \right| \right) \left\langle \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(|\xi_i\rangle \langle \eta_i|) \right\rangle = s \left(X_{l, \xi_i, |\eta_i\rangle \langle \eta_i|}^{(\sigma)} \right) \leq \tau_{y_\sigma} \left(s \left(\omega_{\xi_i} |_{\mathcal{A}_{\sigma, l}} \right) \right) \leq \tau_{y_\sigma} \left(s \left(\omega_\sigma |_{\mathcal{A}_{\sigma, l}} \right) \right).$$

where $y_R = 0$ and $y_L = l$. Hence each term in the decomposition $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) = \sum_{i=1}^{m_\sigma} c_i \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(|\xi_i\rangle \langle \eta_i|)$ is in $\tau_{y_\sigma} \left(s \left(\omega_\sigma |_{\mathcal{A}_{\sigma, l}} \right) \right) \left(\bigotimes_{i=0}^{l-1} \mathbb{C}^n \right)$. From this, we obtain (21).

Next we show

$$0 \leq X_{l, \xi, (1-s(\rho_\sigma))}^{(\sigma)} \leq \left\| T_{\mathbf{v}_\sigma}^l \left(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}} \right) \right\|, \quad \text{for all } \xi \in \mathcal{K}_\sigma, \quad \text{with } \|\xi\| = 1. \quad (22)$$

The first inequality is already proven in Lemma 5.3. The second one follows from the following calculation for any $\zeta \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$:

$$\begin{aligned} \left\langle \zeta, X_{l, \xi, (1-s(\rho_\sigma))}^{(\sigma)} \zeta \right\rangle &= \left\| (1-s(\rho_\sigma)) \left(\sum_{\nu^{(l)}} \left\langle \widehat{\psi_{\nu^{(l)}}}, \zeta \right\rangle \widehat{v_{\nu^{(l), \sigma}}} \right)^* \xi \right\|^2 \leq \left(\sum_{\nu^{(l)}} \left| \left\langle \widehat{\psi_{\nu^{(l)}}}, \zeta \right\rangle \right| \left\| (1-s(\rho_\sigma)) \widehat{v_{\nu^{(l), \sigma}}} \right\| \|\xi\| \right)^2 \\ &\leq \left(\sum_{\nu^{(l)}} \left| \left\langle \widehat{\psi_{\nu^{(l)}}}, \zeta \right\rangle \right|^2 \right) \left(\sum_{\nu^{(l)}} \left\| (1-s(\rho_\sigma)) \widehat{v_{\nu^{(l), \sigma}}} \right\|^2 \right) = \|\zeta\|^2 \left\langle \xi, (T_{\mathbf{v}_\sigma}^l (1-s(\rho_\sigma))) \xi \right\rangle \\ &= \|\zeta\|^2 \left\langle \xi, \left(T_{\mathbf{v}_\sigma}^l \circ (\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}}) (1-s(\rho_\sigma)) + T_{\mathbf{v}_\sigma}^l \circ P_{\{1\}}^{T_{\mathbf{v}_\sigma}} (1-s(\rho_\sigma)) \right) \xi \right\rangle \\ &= \|\zeta\|^2 \left\langle \xi, \left(T_{\mathbf{v}_\sigma}^l \circ (\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}}) (1-s(\rho_\sigma)) \right) \xi \right\rangle \leq \left\| T_{\mathbf{v}_\sigma}^l \circ (\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}}) \right\| \|\zeta\|^2 \|\xi\|^2. \end{aligned}$$

We used Lemma 3.2 in the last equality.

Finally, we claim

$$\text{Ran} \left(X_{l, \xi, s(\rho_\sigma)}^{(\sigma)} \right) \subset \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(B(\mathcal{K}_\sigma) s(\rho_\sigma)), \quad \text{for all } \xi \in \mathcal{K}_\sigma, \quad \text{with } \|\xi\| = 1. \quad (23)$$

To see this, decompose $s(\rho_\sigma)$ as $s(\rho_\sigma) = \sum_i |x_i\rangle \langle x_i|$ with CONS $\{x_i\}$ of $s(\rho_\sigma) \mathcal{K}_\sigma$. We then obtain

$$\begin{aligned} X_{l, \xi, s(\rho_\sigma)}^{(\sigma)} &:= \sum_{\mu^{(l)}, \nu^{(l)}} \left\langle \xi, \widehat{v_{\mu^{(l), \sigma}}} s(\rho_\sigma) \widehat{v_{\nu^{(l), \sigma}}}^* \xi \right\rangle \left| \widehat{\psi_{\mu^{(l)}}} \right\rangle \left\langle \widehat{\psi_{\nu^{(l)}}} \right| = \sum_i \sum_{\mu^{(l)}, \nu^{(l)}} \left\langle \xi, \widehat{v_{\mu^{(l), \sigma}}} (|x_i\rangle \langle x_i|) \widehat{v_{\nu^{(l), \sigma}}}^* \xi \right\rangle \left| \widehat{\psi_{\mu^{(l)}}} \right\rangle \left\langle \widehat{\psi_{\nu^{(l)}}} \right| \\ &= \sum_i \left| \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(|\xi\rangle \langle x_i|) \right\rangle \left\langle \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(|\xi\rangle \langle x_i|) \right\rangle, \end{aligned}$$

for any $\xi \in \mathcal{K}_\sigma$, with $\|\xi\| = 1$. As each $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(|\xi\rangle \langle x_i|)$ is in $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(B(\mathcal{K}_\sigma) s(\rho_\sigma))$, this proves the claim.

Now we derive the claim of the Lemma, combining the above statements. Let ρ_{ω_σ} be the density matrix given by Lemma 2.4. It can be decomposed as $\rho_{\omega_\sigma} = \sum_i \lambda_i |\eta_i\rangle \langle \eta_i|$, with numbers $\lambda_i > 0$, $\sum_i \lambda_i = 1$, and CONS $\{\eta_i\}$ of \mathcal{K}_σ . Then we have

$$\begin{aligned} C_\sigma &\leq \omega_\sigma \left(\tau_{-y_\sigma} \left(\left| \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) \right\rangle \left\langle \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) \right| \right) \right) \\ &= \sum_i \lambda_i \left(\left\langle \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X), X_{l, \eta_i, s(\rho_\sigma)}^{(\sigma)} \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) \right\rangle + \left\langle \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X), X_{l, \eta_i, (1-s(\rho_\sigma))}^{(\sigma)} \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) \right\rangle \right). \end{aligned}$$

In the first inequality, we used (21) and the definition of C_σ . The equality follows from Lemma 2.4, and the definition of $X_{l, \xi, p}^{(\sigma)}$. By the third claim (23) and the orthogonality of $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X)$ and $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(B(\mathcal{K}_\sigma s(\rho_\sigma)))$, the first term on the right hand side is 0. The second term can be bounded using (22) and we have

$$C_\sigma \leq \left\| T_{\mathbf{v}_\sigma}^l \left(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}} \right) \right\| \leq \frac{1}{2} C_\sigma. \quad (24)$$

This is a contradiction. Hence we proved the Lemma. \square

Proof of Lemma 5.1. Set $c_\sigma := \inf\{\sigma(\bar{\rho}_\sigma) \setminus \{0\}\} > 0$. By the routine calculation from Part I, we obtain

$$\left\| \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) \right\|^2 - \rho_\sigma(X^*X) \leq \left\| T_{\mathbf{v}_\sigma}^l \left(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}} \right) \right\| m_\sigma^2 c_\sigma^{-1} \rho_\sigma(X^*X), \quad X \in B(\mathcal{K}_\sigma)s(\rho_\sigma).$$

We set

$$l'_\sigma := \min \left\{ \tilde{l}_\sigma \leq l \in \mathbb{N} \mid \sup_{l' : l \leq l'} \left\{ \left\| T_{\mathbf{v}_\sigma}^{l'} \left(\mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}_\sigma}} \right) \right\| m_\sigma^2 c_\sigma^{-1} \right\} < \frac{1}{2} \right\}$$

with \tilde{l}_σ given in the previous Lemma. Then for $l'_\sigma \leq l$, we have

$$\frac{c_\sigma}{2} \text{Tr}(X^*X) \leq \frac{1}{2} \rho_\sigma(X^*X) \leq \left\| \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(X) \right\|^2, \quad X \in B(\mathcal{K}_\sigma)s(\rho_\sigma).$$

This means $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)} \Big|_{B(\mathcal{K}_\sigma)s(\rho_\sigma)}$ is injective for $l'_\sigma \leq l$. As we have chosen l'_σ so that $\tilde{l}_\sigma \leq l'_\sigma$, it is also onto $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(B(\mathcal{K}_\sigma))$. That $\Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)}(B(\mathcal{K}_\sigma)) = \tau_{y_\sigma} \left(s \left(\omega_\sigma|_{\mathcal{A}_{\sigma, l}} \right) \right) \otimes_{i=0}^{l-1} \mathbb{C}^n$ is proven in Lemma 5.3. \square

As a result, we obtain the following Lemma.

Lemma 5.5. Assume [A1], [A3], [A4], and [A5]. Let $\sigma = L, R$. Let \mathbf{v}_σ be the n -tuple of elements in $B(\mathcal{K}_\sigma)$ given in Notation 2.5, ρ_σ the state given in Lemma 3.2, and $V_{a\sigma}$ the unitary given in Lemma 4.1. Let $l'_\sigma \in \mathbb{N}$ be the number given in Lemma 5.1. For each $a = 0, \dots, k_\sigma$, let $\{g_\alpha^{(a)}\}_{\alpha=1}^{n_0^{(\sigma)}}$ be a CONS of $r_{a\sigma}\mathcal{K}_\sigma \simeq \mathbb{C}^{n_0^{(\sigma)}}$ given by $g_\alpha^{(a)} := V_{a\sigma}\chi_\alpha^{(n_0^{(\sigma)})}$. Then there exist a $y_{a, \alpha, \beta, \sigma}^{(l)} \in B(\mathcal{K}_\sigma)$, for each $a = 0, \dots, k_\sigma$, $\alpha, \beta = 1, \dots, n_0^{(\sigma)}$, and $l \geq l'_\sigma$, satisfying the followings.

(1) For each $l \geq l'_\sigma$, the set $\{y_{a, \alpha, \beta, \sigma}^{(l)}\}_{a=0, \dots, k_\sigma, \alpha, \beta=1, \dots, n_0^{(\sigma)}}$ is a basis of $\mathcal{K}_l(\mathbf{v}_\sigma)$.

(2) For any $a_1 = 0, \dots, k_\sigma$, $\alpha_1, \alpha_2, \beta_1, \beta_2 = 1, \dots, n_0^{(\sigma)}$, and $l_1, l_2 \geq l'_\sigma$, we have

$$y_{a_1, \alpha_1, \beta_1, \sigma}^{(l_1)} y_{a_2, \alpha_2, \beta_2, \sigma}^{(l_2)} = \delta_{\beta_1 \alpha_2} y_{a_1, \alpha_1, \beta_2, \sigma}^{(l_1 + l_2)}.$$

(3) For any $a = 0, \dots, k_\sigma$, $\alpha, \beta = 1, \dots, n_0^{(\sigma)}$, and $l \geq l'_\sigma$,

$$y_{a, \alpha, \beta, \sigma}^{(l)} s(\rho_\sigma) = \left| g_\alpha^{(a)} \right\rangle \left\langle g_\beta^{(0)} \right|.$$

(4) If $X \in \mathcal{K}_l(\mathbf{v}_\sigma)$, $l \geq l'_\sigma$, satisfies $Xs(\rho_\sigma) = 0$, then $X = 0$.

Proof. By Lemma 5.1, for $l \geq l'_\sigma$, the set $\left\{ \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)} \left(\left| g_\alpha^{(a)} \right\rangle \left\langle g_\beta^{(0)} \right| \right) \right\}_{a=0, \dots, k_\sigma, \alpha, \beta=1, \dots, n_0^{(\sigma)}}$ is linearly independent. Therefore, there exist $\xi_{a, \alpha, \beta}^{(l)} \in \otimes_{i=0}^{l-1} \mathbb{C}^n$, $a = 0, \dots, k_\sigma$, $\alpha, \beta = 1, \dots, n_0^{(\sigma)}$ such that

$$\left\langle \xi_{a, \alpha, \beta}^{(l)}, \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)} \left(\left| g_{\alpha'}^{(a')} \right\rangle \left\langle g_{\beta'}^{(0)} \right| \right) \right\rangle = \delta_{aa'} \delta_{\alpha\alpha'} \delta_{\beta\beta'}.$$

Set

$$y_{a, \alpha, \beta, \sigma}^{(l)} := \sum_{\mu^{(l)}} \left\langle \widehat{\psi_{\mu^{(l)}}}, \xi_{a, \alpha, \beta}^{(l)} \right\rangle \widehat{v_{\mu^{(l)}, \sigma}} \in \mathcal{K}_l(\mathbf{v}_\sigma), \quad a = 0, \dots, k_\sigma, \quad \alpha, \beta = 1, \dots, n_0^{(\sigma)}, \quad l \geq l'_\sigma.$$

By a straightforward calculation, we have

$$\delta_{aa'} \delta_{\alpha\alpha'} \delta_{\beta\beta'} = \left\langle \xi_{a, \alpha, \beta}^{(l)}, \Gamma_{l, \mathbf{v}_\sigma}^{(\sigma)} \left(\left| g_{\alpha'}^{(a')} \right\rangle \left\langle g_{\beta'}^{(0)} \right| \right) \right\rangle = \overline{\left\langle g_{\alpha'}^{(a')}, y_{a, \alpha, \beta, \sigma}^{(l)} g_{\beta'}^{(0)} \right\rangle}.$$

This means

$$y_{a,\alpha,\beta,\sigma}^{(l)} s(\rho_\sigma) = \left| g_\alpha^{(a)} \right\rangle \left\langle g_\beta^{(0)} \right|, \quad a = 0, \dots, k_\sigma, \quad \alpha, \beta = 1, \dots, n_0^{(\sigma)}, \quad l \geq l'_\sigma, \quad (25)$$

corresponding to (3) in the claim. This means $\{y_{a,\alpha,\beta,\sigma}^{(l)}\}_{a=0,\dots,k_\sigma,\alpha,\beta=1,\dots,n_0^{(\sigma)}}$ is linearly independent, spanning $(n_0^{(\sigma)})^2(k_\sigma + 1)$ -dimensional subspace of $\mathcal{K}_l(\mathbf{v}_\sigma)$. However, from Lemma 5.1, the dimension of $\mathcal{K}_l(\mathbf{v}_\sigma)$ is $(n_0^{(\sigma)})^2(k_\sigma + 1)$. Hence, $\{y_{a,\alpha,\beta,\sigma}^{(l)}\}_{a=0,\dots,k_\sigma,\alpha,\beta=1,\dots,n_0^{(\sigma)}}$ is a basis of $\mathcal{K}_l(\mathbf{v}_\sigma)$, proving (1).

Let us prove (4). Let $X \in \mathcal{K}_l(\mathbf{v}_\sigma)$, $l \geq l'_\sigma$, such that $Xs(\rho_\sigma) = 0$. Then X can be written as a linear combination $X = \sum_{a\alpha\beta} C_{a\alpha\beta} y_{a,\alpha,\beta,\sigma}^{(l)}$, and we obtain

$$0 = Xs(\rho_\sigma) = \sum_{a\alpha\beta} C_{a\alpha\beta} \left| g_\alpha^{(a)} \right\rangle \left\langle g_\beta^{(0)} \right|.$$

This implies $C_{a\alpha\beta} = 0$ and we conclude $X = 0$.

To prove (2), note that

$$X = y_{a_1,\alpha_1,\beta_1,\sigma}^{(l_1)} y_{0,\alpha_2,\beta_2,\sigma}^{(l_2)} - \delta_{\beta_1\alpha_2} y_{a_1,\alpha_1,\beta_2,\sigma}^{(l_1+l_2)} \in \mathcal{K}_{l_1+l_2}(\mathbf{v}_\sigma)$$

and

$$Xs(\rho_\sigma) = y_{a_1,\alpha_1,\beta_1,\sigma}^{(l_1)} \left| g_{\alpha_2}^{(0)} \right\rangle \left\langle g_{\beta_2}^{(0)} \right| - \delta_{\beta_1\alpha_2} \left| g_{\alpha_1}^{(a_1)} \right\rangle \left\langle g_{\beta_2}^{(0)} \right| = \left| g_{\alpha_1}^{(a_1)} \right\rangle \left\langle g_{\beta_1}^{(0)} \right| \cdot \left| g_{\alpha_2}^{(0)} \right\rangle \left\langle g_{\beta_2}^{(0)} \right| - \delta_{\beta_1\alpha_2} \left| g_{\alpha_1}^{(a_1)} \right\rangle \left\langle g_{\beta_2}^{(0)} \right| = 0.$$

Applying the above argument, we conclude $X = 0$, proving (2). \square

6 Deformation of \mathbf{v}_σ

By Lemma 4.1, we have $n_0^{(\sigma)} = \text{rank } r_{a\sigma} = \text{rank } s(\rho_\sigma)$, for all $a = 0, \dots, k_\sigma$. This means that we can identify $B(\mathcal{K}_\sigma)$ with $M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1}$. We introduce two conditions.

Definition 6.1. Let $n, n_0 \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Let $\omega = (\omega_\mu)_{\mu=1}^n \in M_{n_0}^{\times n}$, $\mathbf{v} = (v_\mu)_{\mu=1}^n \in (M_{n_0} \otimes M_{k+1})^{\times n}$, and $\lambda = (\lambda_a)_{a=0}^k \in \mathbb{C}^{k+1}$. Let $l_0 \in \mathbb{N}$ and $y_{a,\alpha,\beta}^{(l)} \in M_{n_0} \otimes M_{k+1}$, for $a = 0, \dots, k$, $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$. We say that the septuplet $(n_0, k, \omega, \mathbf{v}, \lambda, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 5* if the following holds.

- (i) $\lambda_0 = 1$ and $0 < |\lambda_a| < 1$ for all $a \geq 1$.
- (ii) $v_\mu \in M_{n_0} \otimes \text{DT}_{k+1}$, $\mu = 1, \dots, n$.
- (iii) $(\mathbb{I} \otimes E_{aa}^{(0,k)}) v_\mu (\mathbb{I} \otimes E_{aa}^{(0,k)}) = \lambda_a \omega_\mu \otimes E_{aa}^{(0,k)}$, for all $a = 0, \dots, k$, and $\mu = 1, \dots, n$.
- (iv) (1) For each $l \geq l_0$, the set $\{y_{a,\alpha,\beta}^{(l)}\}_{a=0,\dots,k,\alpha,\beta=1,\dots,n_0}$ is a basis of $\mathcal{K}_l(\mathbf{v})$.
 (2) For any $a_1 = 0, \dots, k$, $\alpha_1, \alpha_2, \beta_1, \beta_2 = 1, \dots, n_0$, and $l_1, l_2 \geq l_0$, we have

$$y_{a_1,\alpha_1,\beta_1}^{(l_1)} y_{0,\alpha_2,\beta_2}^{(l_2)} = \delta_{\beta_1\alpha_2} y_{a_1,\alpha_1,\beta_2}^{(l_1+l_2)}$$

- (3) For any $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$,

$$y_{0,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) y_{0,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right).$$

Definition 6.2. Let $n, n_0 \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$. Let $\omega = (\omega_\mu)_{\mu=1}^n \in M_{n_0}^{\times n}$, $\mathbf{v} = (v_\mu)_{\mu=1}^n \in (M_{n_0} \otimes M_{k+1})^{\times n}$, and $\lambda = (\lambda_a)_{a=0}^k \in \mathbb{C}^{k+1}$. Let $l_0 \in \mathbb{N}$ and $y_{a,\alpha,\beta}^{(l)} \in M_{n_0} \otimes M_{k+1}$, for $a = 0, \dots, k$, $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$. Let $i \in \{0, \dots, k\}$. We say that the septuplet $(n_0, k, \omega, \mathbf{v}, \lambda, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 6-i* if the followings hold.

- (i) The septuplet $(n_0, k, \omega, \mathbf{v}, \lambda, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 5*.
- (ii) There exists a $Y \in \text{DT}_{0,k+1}$ such that $[\Lambda_\lambda, Y] = 0$.
- (iii) For any $\alpha, \beta = 1, \dots, n_0$, $l \geq l_0$, and Y in (ii), we have

$$y_{0,\alpha,\beta}^{(l)} - e_{\alpha\beta}^{(n_0)} \otimes \Lambda_\lambda^l (1 + Y)^l \in M_{n_0} \otimes \sum_{a,a': a-a' \geq i+1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}.$$

Remark 6.3. When we would like to specify Y , we say the septuplet $(n_0, k, \omega, \mathbf{v}, \lambda, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 6-i* with respect to Y . We may set $Y = 0$ for *Condition 6-0*.

Out of our \mathbf{v}_σ , we can construct a septuplet satisfying *Condition 6-0*.

Lemma 6.4. Assume $[A1], [A3], [A4]$, and $[A5]$. We use Notation 2.5 and Notation 3.6. Then for each $\sigma = R, L$, there exist $\omega^{(\sigma)} \in \text{Prim}_u(n, n_0^{(\sigma)})$, $\mathbf{v}^{(\sigma)} \in \left(M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1}\right)^{\times n}$, $\lambda^{(\sigma)} = (\lambda_a^{(\sigma)})_{a=0, \dots, k_\sigma} \in \mathbb{C}^{k_\sigma+1}$, $l_0^{(\sigma)} \in \mathbb{N}$, and $\{y_{a,\alpha,\beta}^{(l,\sigma)}\}_{a=0, \dots, k_\sigma, \alpha, \beta=1, \dots, n_0^{(\sigma)}, l \geq l_0^{(\sigma)}} \subset M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1}$ satisfying the followings.

1. The septuplet $(n_0^{(\sigma)}, k_\sigma, \omega^{(\sigma)}, \mathbf{v}^{(\sigma)}, \lambda^{(\sigma)}, l_0^{(\sigma)}, \{y_{a,\alpha,\beta}^{(l,\sigma)}\})$ satisfies *Condition 6-0*.
2. For the state ω_σ in $[A4]$ and $l \in \mathbb{N}$, $\tau_{y_\sigma}(s(\omega_\sigma|_{\mathcal{A}_{\sigma,l}}))$ is equal to the orthogonal projection onto $\Gamma_{l, \mathbf{v}^{(\sigma)}}^{(\sigma)}(M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1})$, where $y_R = 0$ and $y_L = l$.
3. The triple $(M_{n_0^{(\sigma)}}, \omega^{(\sigma)}, \rho_\sigma|_{M_{n_0^{(\sigma)}}})$ σ -generates ω_∞ .
4. There exist strictly positive elements $h_{a\sigma}$ $a = 0, \dots, k$ in $M_{n_0^{(\sigma)}}$ with $h_{0\sigma} = \mathbb{I}$ such that for all $l \geq l_0^{(\sigma)}$, $a = 0, \dots, k_\sigma$, and $\alpha, \beta = 1, \dots, n_0^{(\sigma)}$,

$$y_{a,\alpha,\beta}^{(l,\sigma)} \left(\mathbb{I} \otimes E_{00}^{(0,k_\sigma)} \right) = h_{a\sigma}^{\frac{1}{2}} e_{\alpha\beta}^{(n_0^{(\sigma)})} \otimes E_{a0}^{(0,k_\sigma)}.$$

Proof. We use Notation 2.5 and Notation 3.6. Define $\omega^{(\sigma)} := \widetilde{\mathbf{v}}_\sigma \in M_{n_0^{(\sigma)}}^{\times n}$ under the identification $M_{n_0^{(\sigma)}} \simeq B(s(\rho_\sigma)\mathcal{K}_\sigma)$. By Lemma 3.3, we have $\omega^{(\sigma)} \in \text{Prim}_u(n, n_0)$, and $(M_{n_0^{(\sigma)}}, \omega^{(\sigma)}, \rho_\sigma|_{M_{n_0^{(\sigma)}}})$ σ -generates ω_∞ . (This proves 3 of the Lemma.)

From Notation 3.6 and Lemma 4.1, for each $a = 0, \dots, k_\sigma$, there is a unitary $V_{a\sigma} : \mathbb{C}^{n_0^{(\sigma)}} \rightarrow r_{a\sigma}\mathcal{K}_\sigma$ and $c_{a\sigma} \in \mathbb{T}$ such that

$$r_{T_{\omega_{a,\sigma}}}^{-\frac{1}{2}} t_{a\sigma}^{-\frac{1}{2}} r_{a\sigma} v_{\mu\sigma} r_{a\sigma} t_{a\sigma}^{\frac{1}{2}} = u_{\mu a\sigma} = c_{a\sigma} V_{a\sigma} \widetilde{v_{\mu\sigma}} V_{a\sigma}^* = c_{a\sigma} V_{a\sigma} \omega_\mu^{(\sigma)} V_{a\sigma}^*, \quad \mu = 1, \dots, n. \quad (26)$$

Note that $r_{T_{\omega_{0,\sigma}}} = 1$, and we may choose $c_{0\sigma} = 1$, $t_{0\sigma} = r_{0\sigma}$, and $V_{0\sigma}$ is the identity map.

Define an invertible element $R^{(\sigma)} := \bigoplus_{a=0}^{k_\sigma} t_{a\sigma}^{\frac{1}{2}}$ in $B(\mathcal{K}_\sigma)$, and $\lambda^{(\sigma)} = (\lambda_a^{(\sigma)})_{a=0, \dots, k_\sigma} \in \mathbb{C}^{k_\sigma+1}$ by $\lambda_a^{(\sigma)} = r_{T_{\omega_{a,\sigma}}}^{\frac{1}{2}} c_{a\sigma}$. Furthermore, we set $h_{a\sigma}^{\frac{1}{2}} := V_{a\sigma}^* t_{a\sigma}^{-\frac{1}{2}} V_{a\sigma}$. Note that $h_{0\sigma} = 1$. Using unitaries $V_{a\sigma}$ above, we define a linear map $V^\sigma : \mathcal{K}_\sigma \rightarrow \mathbb{C}^{n_0^{(\sigma)}} \otimes \mathbb{C}^{k_\sigma+1}$ by

$$V^{(\sigma)} \xi := \sum_{a=0}^{k_\sigma} (V_{a\sigma}^* r_{a\sigma} \xi) \otimes f_a^{(0,k_\sigma)}, \quad \xi \in \mathcal{K}_\sigma.$$

By definition, this is unitary.

Let l'_σ , and $\{y_{a,\alpha,\beta,\sigma}^{(l)}\}_{a=0,\dots,k_\sigma, \alpha,\beta=1,\dots,n_0^{(\sigma)}, l \geq l'_\sigma} \subset B(\mathcal{K}_\sigma)$ given in Lemma 5.5.

We define $\mathbf{v}^{(\sigma)} = (v_\mu^{(\sigma)})_{\mu=1}^n \in \left(M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1}\right)^{\times n}$ by

$$v_\mu^{(\sigma)} := V^{(\sigma)}(R^{(\sigma)})^{-1}v_{\mu\sigma}R^{(\sigma)}V^{(\sigma)*}, \quad \mu = 1, \dots, n.$$

We also set

$$y_{a,\alpha,\beta}^{(l,\sigma)} = V^{(\sigma)}(R^{(\sigma)})^{-1}y_{a,\alpha,\beta,\sigma}^{(l)}R^{(\sigma)}V^{(\sigma)*}, \quad a = 0, \dots, k_\sigma, \alpha, \beta = 1, \dots, n_0^{(\sigma)}, l \geq l'_\sigma.$$

Furthermore, we set $l_0^{(\sigma)} := l'_\sigma$. As $v_\mu^{(\sigma)}$ is similar to $v_{\mu\sigma}$ with common invertible operator $R^{(\sigma)}V^{(\sigma)*}$, Lemma 5.3 implies 2. of the current Lemma.

Next we show that the septuplet $(n_0^{(\sigma)}, k_\sigma, \boldsymbol{\omega}^{(\sigma)}, \mathbf{v}^{(\sigma)}, \boldsymbol{\lambda}^{(\sigma)}, l_0^{(\sigma)}, \{y_{a,\alpha,\beta}^{(l,\sigma)}\})$ satisfies *Condition 5*. (i) follows from Lemma 3.7 and the above remark on $c_{0\sigma}$ etc. Recall the definition of $\mathcal{K}_{a,\sigma}$, given as an v_ν^* -invariant subspace. This property is translated to (ii) of *Condition 5* for $\mathbf{v}^{(\sigma)}$. The equality (26) implies (iii) of *Condition 5*. (1), (2) of (iv) follows from the fact that $y_{a,\alpha,\beta}^{(l,\sigma)}$ (resp. $\mathbf{v}^{(\sigma)}$) and $y_{a,\alpha,\beta,\sigma}^{(l)}$ (resp. \mathbf{v}_σ) are similar to each other with common invertible operator $R^{(\sigma)}V^{(\sigma)*}$. (3) of (iv) follows from (3) of Lemma 5.5 and the fact that $R^{(\sigma)}V^{(\sigma)*}(\mathbb{I} \otimes E_{00}^{(0,k_\sigma)}) = s(\rho_\sigma)R^{(\sigma)}V^{(\sigma)*}$.

It is left to prove (ii), (iii) of *Condition 6-0*. We set $Y = 0$. Then clearly (ii) holds. From (ii), (iii) of *Condition 5* which we have already proved, we see that

$$(\mathbb{I} \otimes E_{aa}^{(0,k)})v_{\mu^{(l)}}^{(\sigma)}(\mathbb{I} \otimes E_{aa}^{(0,k)}) = \left(\lambda_a^{(\sigma)}\right)^l \omega_{\mu^{(l)}}^{(\sigma)} \otimes E_{aa}^{(0,k)}, \quad \mu^{(l)} \in \{1, \dots, n\}^{\times l}, \quad l \in \mathbb{N}, \quad a = 0, \dots, k.$$

From this, we obtain

$$(\mathbb{I} \otimes E_{aa}^{(0,k)})y_{0,\alpha,\beta}^{(l,\sigma)}(\mathbb{I} \otimes E_{aa}^{(0,k)}) = \left(\lambda_a^{(\sigma)}\right)^l e_{\alpha\beta}^{(n_0)} \otimes E_{aa}^{(0,k)}.$$

As $y_{a,\alpha,\beta}^{(l,\sigma)}$ is a lower triangular matrix (because each $v_\mu^{(\sigma)}$ is), this implies (iii) of *Condition 6-0*.

Lastly, we prove 4. As we observed, we have

$$R^{(\sigma)}V^{(\sigma)*}(\mathbb{I} \otimes E_{00}^{(0,k_\sigma)}) = s(\rho_\sigma)R^{(\sigma)}V^{(\sigma)*}, \quad (\mathbb{I} \otimes E_{00}^{(0,k_\sigma)})V^{(\sigma)}R^{(\sigma)} = V^{(\sigma)}R^{(\sigma)}s(\rho_\sigma).$$

Therefore, we have

$$\begin{aligned} y_{a,\alpha,\beta}^{(l,\sigma)}(\mathbb{I} \otimes E_{00}^{(0,k_\sigma)}) &= V^{(\sigma)}(R^{(\sigma)})^{-1}y_{a,\alpha,\beta,\sigma}^{(l)}R^{(\sigma)}V^{(\sigma)*}(\mathbb{I} \otimes E_{00}^{(0,k_\sigma)}) = V^{(\sigma)}(R^{(\sigma)})^{-1}y_{a,\alpha,\beta,\sigma}^{(l)}s(\rho_\sigma)R^{(\sigma)}V^{(\sigma)*} \\ &= V^{(\sigma)}t_{a\sigma}^{-\frac{1}{2}}\left|g_\alpha^{(a)}\right\rangle\left\langle g_\beta^{(0)}\right| R^{(\sigma)}V^{(\sigma)*} = \left(h_{a\sigma}^{\frac{1}{2}} \otimes E_{aa}^{(0,k_\sigma)}\right)V^{(\sigma)}\left|g_\alpha^{(a)}\right\rangle\left\langle g_\beta^{(0)}\right| R^{(\sigma)}V^{(\sigma)*} = h_{a\sigma}^{\frac{1}{2}}e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k_\sigma)}. \end{aligned}$$

□

We prove that *Condition 6-0* implies *Condition 6- k_σ* , inductively.

Lemma 6.5. *Let $0 \leq i \leq k-1$. Assume that the septuplet $(n_0, k, \boldsymbol{\omega}, \mathbf{v}, \boldsymbol{\lambda}, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 6- i* with respect to Y . Then there exist $\{J_j\}_{j=i+1}^k \subset M_{n_0}$ and $\{c_j\}_{j=i+1}^k \subset \mathbb{C}$ satisfying the followings.*

1. Set $R := \mathbb{I} - \sum_{j=i+1}^k J_j \otimes E_{j,j-(i+1)}^{(0,k)}$. Then R is invertible, $R - \mathbb{I} \in M_{n_0} \otimes \text{DT}_{0,k+1}$ and $R(\mathbb{I} \otimes E_{00}^{(0,k)}) = R^{-1}(\mathbb{I} \otimes E_{00}^{(0,k)}) = \mathbb{I} \otimes E_{00}^{(0,k)}$.

2. Set $Y' := \sum_{i+1 \leq j \leq k: \lambda_j = \lambda_{j-(i+1)}} c_j E_{j, j-(i+1)}^{(0,k)}$. The septuplet $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{Ry_{a, \alpha, \beta}^{(l)} R^{-1}\})$ satisfies Condition 6-(i+1) with respect to $Y + Y'$.

For the proof, we need the following Lemma.

Lemma 6.6. Let $n_0, l_0 \in \mathbb{N}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and suppose that $x_{\alpha, \beta}^{(l)} \in M_{n_0}$, $\alpha, \beta = 1, \dots, n_0$, are given for all $l \geq l_0$. Assume that they satisfy the following condition: For any $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \{1, \dots, n_0\}$ and $l_1, l_2 \geq l_0$,

$$x_{\alpha_1, \beta_1}^{(l_1)} e_{\alpha_2 \beta_2}^{(n_0)} + \lambda^{-l_1} e_{\alpha_1 \beta_1}^{(n_0)} x_{\alpha_2 \beta_2}^{(l_2)} = \delta_{\beta_1, \alpha_2} x_{\alpha_1, \beta_2}^{(l_1 + l_2)}. \quad (27)$$

Then we have the followings.

- (1) If $\lambda \neq 1$, there exists $J \in M_{n_0}$ such that

$$x_{\alpha \beta}^{(l)} = J e_{\alpha \beta}^{(n_0)} - \lambda^{-l} e_{\alpha \beta}^{(n_0)} J, \quad l \geq l_0, \quad \alpha, \beta \in \{1, \dots, n_0\}.$$

- (2) If $\lambda = 1$, there exist $J \in M_{n_0}$ and $c \in \mathbb{C}$ such that

$$x_{\alpha \beta}^{(l)} = J e_{\alpha \beta}^{(n_0)} - e_{\alpha \beta}^{(n_0)} J + c \cdot l \cdot e_{\alpha \beta}^{(n_0)}, \quad l \geq l_0, \quad \alpha, \beta \in \{1, \dots, n_0\}.$$

Proof. We claim there exists $\tilde{J} \in M_{n_0}$ such that

$$x_{\alpha \beta}^{(l)} (1 - e_{\beta \beta}^{(n_0)}) = -\lambda^{-l} e_{\alpha \beta}^{(n_0)} \tilde{J}, \quad (1 - e_{\alpha \alpha}^{(n_0)}) x_{\alpha \beta}^{(l)} = \tilde{J} e_{\alpha \beta}^{(n_0)}, \quad l \geq l_0, \quad \alpha, \beta \in \{1, \dots, n_0\}. \quad (28)$$

To see this, set $F_\beta := \sum_{\alpha: \alpha \neq \beta} x_{\alpha \alpha}^{(l_0)}$. Then by (27) with $\beta_1 \neq \alpha_2$, we have for $l \geq l_0$ and $\alpha, \beta \in \{1, \dots, n_0\}$,

$$x_{\alpha \beta}^{(l)} (1 - e_{\beta \beta}^{(n_0)}) + \lambda^{-l} e_{\alpha \beta}^{(n_0)} F_\beta = x_{\alpha \beta}^{(l)} \sum_{\alpha_2: \alpha_2 \neq \beta} e_{\alpha_2 \alpha_2}^{(n_0)} + \lambda^{-l} e_{\alpha \beta}^{(n_0)} \sum_{\alpha_2: \alpha_2 \neq \beta} x_{\alpha_2 \alpha_2}^{(l_0)} = 0. \quad (29)$$

We also have for $l \geq l_0$ and $\alpha, \beta \in \{1, \dots, n_0\}$,

$$F_\alpha e_{\alpha \beta}^{(n_0)} + \lambda^{-l_0} (1 - e_{\alpha \alpha}^{(n_0)}) x_{\alpha \beta}^{(l)} = \sum_{\alpha_1: \alpha_1 \neq \alpha} x_{\alpha_1 \alpha_1}^{(l_0)} e_{\alpha \beta}^{(n_0)} + \sum_{\alpha_1: \alpha_1 \neq \alpha} \lambda^{-l_0} e_{\alpha_1 \alpha_1}^{(n_0)} x_{\alpha \beta}^{(l)} = 0 \quad (30)$$

from (27).

Set $\tilde{J} := \sum_{\beta=1}^{n_0} e_{\beta \beta}^{(n_0)} F_\beta (1 - e_{\beta \beta}^{(n_0)})$ and $\tilde{J}' := -\lambda^{l_0} \sum_{\alpha=1}^{n_0} (1 - e_{\alpha \alpha}^{(n_0)}) F_\alpha e_{\alpha \alpha}^{(n_0)}$. By (29) and (30), we have for $l \geq l_0$ and $\alpha, \beta \in \{1, \dots, n_0\}$,

$$x_{\alpha \beta}^{(l)} (1 - e_{\beta \beta}^{(n_0)}) = -\lambda^{-l} e_{\alpha \beta}^{(n_0)} F_\beta = -\lambda^{-l} e_{\alpha \beta}^{(n_0)} \tilde{J}, \quad (1 - e_{\alpha \alpha}^{(n_0)}) x_{\alpha \beta}^{(l)} = -\lambda^{l_0} F_\alpha e_{\alpha \beta}^{(n_0)} = \tilde{J}' e_{\alpha \beta}^{(n_0)}. \quad (31)$$

To complete the proof of the claim, we show $\tilde{J} = \tilde{J}'$. For any $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \{1, \dots, n_0\}$ with $\beta_1 \neq \alpha_2$ and $l_1, l_2 \geq l_0$, we have from (31) and (27),

$$-\lambda^{-l_1} e_{\alpha_1 \beta_1}^{(n_0)} \tilde{J} e_{\alpha_2 \beta_2}^{(n_0)} + \lambda^{-l_1} e_{\alpha_1 \beta_1}^{(n_0)} \tilde{J}' e_{\alpha_2 \beta_2}^{(n_0)} = x_{\alpha_1 \beta_1}^{(l_1)} e_{\alpha_2 \beta_2}^{(n_0)} + \lambda^{-l_1} e_{\alpha_1 \beta_1}^{(n_0)} x_{\alpha_2 \beta_2}^{(l_2)} = 0.$$

This means the off diagonal elements of \tilde{J} and \tilde{J}' coincide. As the diagonal elements of \tilde{J} and \tilde{J}' are zero, we obtain $\tilde{J} = \tilde{J}'$, proving the claim.

To proceed, we first consider the $\lambda \neq 1$ case. We fix some $l'_0 \geq l_0$ such that $\lambda^{l'_0} \neq 1$. Set

$$C_{\alpha \beta}^{(l)} := \langle \chi_\alpha^{(n_0)}, x_{\alpha \beta}^{(l)} \chi_\beta^{(n_0)} \rangle, \quad l \geq l_0, \quad \alpha, \beta = 1, \dots, n_0,$$

and $d := (1 - \lambda^{-l'_0})^{-1} \lambda^{-l'_0} C_{11}^{(l'_0)}$. We claim

$$C_{\alpha\beta}^{(l)} = C_{\alpha 1}^{(l'_0)} + d - \lambda^{-l} \left(C_{\beta 1}^{(l'_0)} + d \right), \quad l \geq l_0, \quad \alpha, \beta = 1, \dots, n_0. \quad (32)$$

For any $\alpha_1, \beta_2, \beta \in \{1, \dots, n_0\}$ and $l_1, l_2 \geq l_0$, substituting (28), we have

$$\begin{aligned} & x_{\alpha_1, \beta}^{(l_1)} e_{\beta, \beta_2}^{(n_0)} + \lambda^{-l_1} e_{\alpha_1 \beta}^{(n_0)} x_{\beta \beta_2}^{(l_2)} \\ &= e_{\alpha_1 \alpha_1}^{(n_0)} x_{\alpha_1, \beta}^{(l_1)} e_{\beta, \beta_2}^{(n_0)} + \left(1 - e_{\alpha_1 \alpha_1}^{(n_0)}\right) x_{\alpha_1, \beta}^{(l_1)} e_{\beta, \beta_2}^{(n_0)} + \lambda^{-l_1} e_{\alpha_1 \beta}^{(n_0)} x_{\beta \beta_2}^{(l_2)} e_{\beta_2 \beta_2}^{(n_0)} + \lambda^{-l_1} e_{\alpha_1 \beta}^{(n_0)} x_{\beta \beta_2}^{(l_2)} (1 - e_{\beta_2 \beta_2}^{(n_0)}) \\ &= \left(C_{\alpha_1 \beta}^{(l_1)} + \lambda^{-l_1} C_{\beta \beta_2}^{(l_2)} \right) e_{\alpha_1 \beta_2}^{(n_0)} + \tilde{J} e_{\alpha_1 \beta_2}^{(n_0)} - \lambda^{-l_1 - l_2} e_{\alpha_1 \beta_2}^{(n_0)} \tilde{J}. \end{aligned}$$

Note that the left hand side is β -independent because of (27), and the second and the third term on the right hand side is also β -independent. Therefore, for any $\alpha_1, \beta_2 \in \{1, \dots, n_0\}$, and $l_1, l_2 \geq l_0$, $C_{\alpha_1 \beta}^{(l_1)} + \lambda^{-l_1} C_{\beta \beta_2}^{(l_2)}$ is β -independent. Hence, for $\alpha, \beta \in \{1, \dots, n_0\}$, and $l \geq l_0$,

$$C_{\alpha\beta}^{(l)} = -\lambda^{-l} C_{\beta 1}^{(l'_0)} + C_{\alpha 1}^{(l)} + \lambda^{-l} C_{11}^{(l'_0)}. \quad (33)$$

Substituting this to $C_{\alpha_1 \beta}^{(l_1)} + \lambda^{-l_1} C_{\beta \beta_2}^{(l_2)}$, we see that for $\alpha_1, \beta_2, \beta = 1, \dots, n_0$ and $l_1, l_2 \geq l_0$,

$$C_{\alpha_1 \beta}^{(l_1)} + \lambda^{-l_1} C_{\beta \beta_2}^{(l_2)} = -\lambda^{-l_1} C_{\beta 1}^{(l'_0)} + C_{\alpha_1 1}^{(l_1)} + \lambda^{-l_1} C_{11}^{(l'_0)} + \lambda^{-l_1} \left(-\lambda^{-l_2} C_{\beta_2 1}^{(l'_0)} + C_{\beta 1}^{(l_2)} + \lambda^{-l_2} C_{11}^{(l'_0)} \right).$$

Recall that the left hand side is β -independent. This means, $W_l := C_{\beta 1}^{(l)} - C_{\beta 1}^{(l'_0)}$ is β -independent for $l \geq l_0$. Note that $W_{l'_0} = 0$.

Using W_l , (33) can be written as

$$C_{\alpha\beta}^{(l)} = \lambda^{-l} \left(C_{11}^{(l'_0)} - C_{\beta 1}^{(l'_0)} \right) + W_l + C_{\alpha 1}^{(l'_0)}, \quad l \geq l_0, \quad \alpha, \beta = 1, \dots, n_0. \quad (34)$$

Considering (α_1, β_2) -matrix element of (27), with $\beta_1 = \alpha_2 = \beta$, $l_1 = l \geq l_0$ and $l_2 = l_0$, we have $C_{\alpha_1 \beta}^{(l)} + \lambda^{-l} C_{\beta \beta_2}^{(l'_0)} = C_{\alpha_1 \beta_2}^{(l+l'_0)}$. Same consideration with $l_1 = l'_0$ and $l_2 = l \geq l_0$, implies $C_{\alpha_1 \beta}^{(l'_0)} + \lambda^{-l'_0} C_{\beta \beta_2}^{(l)} = C_{\alpha_1 \beta_2}^{(l+l'_0)}$. From these, we obtain $C_{\alpha_1 \beta}^{(l)} + \lambda^{-l} C_{\beta \beta_2}^{(l'_0)} = C_{\alpha_1 \beta}^{(l'_0)} + \lambda^{-l'_0} C_{\beta \beta_2}^{(l)}$. Substituting (34) and $W_{l'_0} = 0$ to this, we obtain

$$W_l = (\lambda^{-l'_0} - \lambda^{-l}) \left(1 - \lambda^{-l'_0} \right)^{-1} C_{1,1}^{(l'_0)}, \quad l \geq l_0.$$

Substituting this to (34), we obtain (32).

Set $J := \tilde{J} + \sum_{\alpha} \left(C_{\alpha 1}^{(l'_0)} + d \right) e_{\alpha \alpha}^{(n_0)} \in M_{n_0}$. Then for $l \geq l_0$ and $\alpha, \beta \in \{1, \dots, n_0\}$, using (28),

$$\begin{aligned} x_{\alpha\beta}^{(l)} &= \left(1 - e_{\alpha\alpha}^{(n_0)} \right) x_{\alpha\beta}^{(l)} + e_{\alpha\alpha}^{(n_0)} x_{\alpha\beta}^{(l)} \left(1 - e_{\beta\beta}^{(n_0)} \right) + e_{\alpha\alpha}^{(n_0)} x_{\alpha\beta}^{(l)} e_{\beta\beta}^{(n_0)} = \tilde{J} e_{\alpha\beta}^{(n_0)} - \lambda^{-l} e_{\alpha\beta}^{(n_0)} \tilde{J} + C_{\alpha\beta}^{(l)} e_{\alpha\beta}^{(n_0)} \\ &= \tilde{J} e_{\alpha\beta}^{(n_0)} - \lambda^{-l} e_{\alpha\beta}^{(n_0)} \tilde{J} + \left(C_{\alpha 1}^{(l'_0)} + d - \lambda^{-l} \left(C_{\beta 1}^{(l'_0)} + d \right) \right) e_{\alpha\beta}^{(n_0)} = J e_{\alpha\beta}^{(n_0)} - \lambda^{-l} e_{\alpha\beta}^{(n_0)} J. \end{aligned}$$

Now we turn to the case $\lambda = 1$. Set

$$C_{\alpha\beta}^{(l)} := \left\langle \chi_{\alpha}^{(n_0)}, x_{\alpha\beta}^{(l)} \chi_{\beta}^{(n_0)} \right\rangle, \quad l \geq l_0, \quad \alpha, \beta = 1, \dots, n_0.$$

By (27) with $\beta_1 = \alpha_2 = \beta$ we have

$$C_{\alpha_1 \beta}^{(l_1)} + C_{\beta, \beta_2}^{(l_2)} = C_{\alpha_1, \beta_2}^{(l_1 + l_2)}, \quad \alpha_1, \beta_2, \beta \in \{1, \dots, n_0\}, \quad l_1, l_2 \geq l_0. \quad (35)$$

Therefore, for any $l \geq l_0$, we have

$$C_{11}^{(l+2)} - C_{11}^{(l+1)} = C_{11}^{(l+1)} - C_{11}^{(l)},$$

and obtain

$$C_{11}^{(l)} - C_{11}^{(l_0)} = \sum_{j=l_0+1}^l (C_{11}^{(j)} - C_{11}^{(j-1)}) = (l - l_0) (C_{11}^{(l_0+1)} - C_{11}^{(l_0)}), \quad l \geq l_0 + 1. \quad (36)$$

Note that we also have $C_{11}^{(l_0)} - C_{11}^{(l_0)} = 0 = (l_0 - l_0) (C_{11}^{(l_0+1)} - C_{11}^{(l_0)})$. Set $c' := -l_0 (C_{11}^{(l_0+1)} - C_{11}^{(l_0)}) + C_{11}^{(l_0)}$ and $c = C_{11}^{(l_0+1)} - C_{11}^{(l_0)}$. Then from (35) and (36), we have

$$\begin{aligned} C_{\alpha\beta}^{(l)} &= C_{11}^{(l_0)} - C_{\beta 1}^{(l_0)} + C_{\alpha 1}^{(l)} \\ &= C_{11}^{(l_0)} - C_{\beta 1}^{(l_0)} + (C_{\alpha 1}^{(l)} - C_{\alpha 1}^{(l_0)}) + C_{\alpha 1}^{(l_0)} \\ &= C_{11}^{(l_0)} - C_{\beta 1}^{(l_0)} + (C_{11}^{(l)} - C_{11}^{(l_0)}) + C_{\alpha 1}^{(l_0)} \\ &= C_{11}^{(l_0)} - C_{\beta 1}^{(l_0)} + (l - l_0) (C_{11}^{(l_0+1)} - C_{11}^{(l_0)}) + C_{\alpha 1}^{(l_0)} \\ &= C_{\alpha 1}^{(l_0)} - C_{\beta 1}^{(l_0)} + c \cdot l + c', \end{aligned}$$

for all $\alpha, \beta \in \{1, \dots, n_0\}$, $l \geq l_0$. Substituting this to (35), we get $c' = 0$. Hence we obtain

$$C_{\alpha\beta}^{(l)} = C_{\alpha 1}^{(l_0)} - C_{\beta 1}^{(l_0)} + c \cdot l, \quad l \geq l_0, \quad \alpha, \beta = 1, \dots, n_0. \quad (37)$$

Set $\tilde{I} := \sum_{\alpha=1}^{n_0} C_{\alpha,1}^{(l_0)} e_{\alpha\alpha}^{(n_0)} \in M_{n_0}$. Recall \tilde{J} in (28), and set $J := \tilde{J} + \tilde{I}$. Then substituting (37), we have for $l \geq l_0$, and $\alpha, \beta = 1, \dots, n_0$,

$$\begin{aligned} x_{\alpha\beta}^{(l)} &= (1 - e_{\alpha\alpha}^{(n_0)}) x_{\alpha\beta}^{(l)} + e_{\alpha\alpha}^{(n_0)} x_{\alpha\beta}^{(l)} (1 - e_{\beta\beta}^{(n_0)}) + e_{\alpha\alpha}^{(n_0)} x_{\alpha\beta}^{(l)} e_{\beta\beta}^{(n_0)} = \tilde{J} e_{\alpha\beta}^{(n_0)} - e_{\alpha\beta}^{(n_0)} \tilde{J} + C_{\alpha\beta}^{(l)} e_{\alpha\beta}^{(n_0)} \\ &= \tilde{J} e_{\alpha\beta}^{(n_0)} - e_{\alpha\beta}^{(n_0)} \tilde{J} + (C_{\alpha 1}^{(l_0)} - C_{\beta 1}^{(l_0)} + c \cdot l) e_{\alpha\beta}^{(n_0)} = \tilde{J} e_{\alpha\beta}^{(n_0)} - e_{\alpha\beta}^{(n_0)} \tilde{J} + \tilde{I} e_{\alpha\beta}^{(n_0)} - e_{\alpha\beta}^{(n_0)} \tilde{I} + c \cdot l \cdot e_{\alpha\beta}^{(n_0)} \\ &= J e_{\alpha\beta}^{(n_0)} - e_{\alpha\beta}^{(n_0)} J + c \cdot l \cdot e_{\alpha\beta}^{(n_0)}. \end{aligned}$$

□

Proof of Lemma 6.5. By Condition 6-i, each $y_{0,\alpha,\beta}^{(l)}$, $l \geq l_0$, and $\alpha, \beta = 1, \dots, n_0$, is of the form

$$y_{0,\alpha,\beta}^{(l)} = e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l + \sum_{j=i+1}^k x_{\alpha,\beta,j}^{(l)} \otimes E_{j,j-(i+1)}^{(0,k)} + Y_{\alpha,\beta}^{(l)},$$

with $x_{\alpha,\beta,j}^{(l)} \in M_{n_0}$ and $Y_{\alpha,\beta}^{(l)} \in M_{n_0} \otimes \sum_{a,a':a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$. Recall that

$$y_{0,\alpha_1,\beta_1}^{(l_1)} y_{0,\alpha_2,\beta_2}^{(l_2)} = \delta_{\beta_1\alpha_2} y_{0,\alpha_1,\beta_2}^{(l_1+l_2)}, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 = 1, \dots, n_0, \quad l_1, l_2 \geq l_0,$$

by Condition 6-i. With the above representation, the left hand side of this equation can be written

$$\begin{aligned} &y_{0,\alpha_1,\beta_1}^{(l_1)} y_{0,\alpha_2,\beta_2}^{(l_2)} \\ &= \delta_{\beta_1\alpha_2} e_{\alpha_1\beta_2}^{(n_0)} \otimes \Lambda_{\lambda}^{l_1+l_2} (\mathbb{I} + Y)^{l_1+l_2} + \sum_{j=i+1}^k \left(\lambda_j^{l_1} e_{\alpha_1\beta_1}^{(n_0)} x_{\alpha_2,\beta_2,j}^{(l_2)} + \lambda_{j-(i+1)}^{l_2} x_{\alpha_1,\beta_1,j}^{(l_1)} e_{\alpha_2\beta_2}^{(n_0)} \right) \otimes E_{j,j-(i+1)}^{(0,k)} \\ &\quad + \text{an element of } M_{n_0} \otimes \sum_{a,a':a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}. \end{aligned}$$

Compare the $M_{n_0} \otimes \sum_{a,a':a-a'=i+1} E_{aa'}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$ part of this and that of the right hand side:

$$\delta_{\beta_1 \alpha_2} y_{0, \alpha_1, \beta_2}^{(l_1+l_2)} = \delta_{\beta_1 \alpha_2} \left(e_{\alpha_1 \beta_2}^{(n_0)} \otimes \Lambda_{\lambda}^{l_1+l_2} (\mathbb{I} + Y)^{l_1+l_2} + \sum_{j=i+1}^k x_{\alpha_1, \beta_2, j}^{(l_1+l_2)} \otimes E_{j, j-(i+1)}^{(0,k)} + Y_{\alpha_1, \beta_2}^{(l_1+l_2)} \right). \quad (38)$$

Then we obtain

$$\lambda_j^{l_1} e_{\alpha_1 \beta_1}^{(n_0)} x_{\alpha_2, \beta_2, j}^{(l_2)} + \lambda_{j-(i+1)}^{l_2} x_{\alpha_1, \beta_1, j}^{(l_1)} e_{\alpha_2 \beta_2}^{(n_0)} = \delta_{\beta_1 \alpha_2} x_{\alpha_1, \beta_2, j}^{(l_1+l_2)}.$$

Set $\tilde{x}_{\alpha, \beta, j}^{(l)} := \lambda_{j-(i+1)}^{-l} x_{\alpha, \beta, j}^{(l)}$. From the above equality, $\tilde{x}_{\alpha, \beta, j}^{(l)}$ satisfies condition of Lemma 6.6 with $\lambda = \lambda_{j-(i+1)}/\lambda_j$. Applying Lemma 6.6, we obtain $\{J_j\}_{j=i+1}^k \subset M_{n_0}$ and $\{c_j\}_{j=i+1}^k \subset \mathbb{C}$ such that

$$x_{\alpha, \beta, j}^{(l)} = \lambda_{j-(i+1)}^l J_j e_{\alpha \beta}^{(n_0)} - \lambda_j^l e_{\alpha \beta}^{(n_0)} J_j + \delta_{\lambda_j, \lambda_{j-(i+1)}} c_j l \lambda_j^l e_{\alpha \beta}^{(n_0)}, \quad \alpha, \beta = 1, \dots, n_0, \quad l \geq l_0. \quad (39)$$

Hence, for any $i+1 \leq j \leq k$, $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$, we have

$$\begin{aligned} & \left(\mathbb{I} \otimes E_{jj}^{(0,k)} \right) \left(y_{0, \alpha, \beta}^{(l)} - e_{\alpha \beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (1 + Y)^l \right) \left(\mathbb{I} \otimes E_{j-(i+1), j-(i+1)}^{(0,k)} \right) \\ &= \left(\lambda_{j-(i+1)}^l J_j e_{\alpha \beta}^{(n_0)} - \lambda_j^l e_{\alpha \beta}^{(n_0)} J_j + \delta_{\lambda_j, \lambda_{j-(i+1)}} c_j l \lambda_j^l e_{\alpha \beta}^{(n_0)} \right) \otimes E_{j, j-(i+1)}^{(0,k)}. \end{aligned} \quad (40)$$

Now we check 1,2 of the Lemma. We start from 1. Set $\hat{J} := \sum_{j=i+1}^k J_j \otimes E_{j, j-(i+1)}^{(0,k)}$, and $R := \mathbb{I} - \hat{J}$. Clearly, $\hat{J} \in M_{n_0} \otimes \text{DT}_{0, k+1}$. As $\hat{J}^{k+1} = 0$, R is invertible and $R^{-1} = \mathbb{I} + \hat{J} + \dots + \hat{J}^k$. Consider (40) with $j = i+1$. Then by Condition 5 (iv) (3) for $\{y_{a, \alpha, \beta}^{(l)}\}$, we get

$$0 = \left(\mathbb{I} \otimes E_{i+1, i+1}^{(0,k)} \right) \left(y_{0, \alpha, \beta}^{(l)} - e_{\alpha \beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (1 + Y)^l \right) \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = \left(J_{i+1} e_{\alpha \beta}^{(n_0)} - \lambda_{i+1}^l e_{\alpha \beta}^{(n_0)} J_{i+1} \right) \otimes E_{i+1, 0}^{(0,k)},$$

for all $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$. This implies $J_{i+1} = 0$. Therefore, we have $\hat{J} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = J_{i+1} \otimes E_{i+1, 0}^{(0,k)} = 0$, which implies

$$R \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = R^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = \mathbb{I} \otimes E_{00}^{(0,k)}. \quad (41)$$

To prove 2, we first show that the septuplet $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{Ry_{a, \alpha, \beta}^{(l)} R^{-1}\})$ satisfies Condition 5. (i) follows from the assumption that Condition 6-i holds. As all of v_μ, R, R^{-1} belongs to $M_{n_0} \otimes \text{DT}_{k+1}$, we have $Rv_\mu R^{-1} \in M_{n_0} \otimes \text{DT}_{k+1}$, proving (ii). (iii) can be checked as follows:

$$\begin{aligned} & \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) Rv_\mu R^{-1} \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) = \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) R \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) v_\mu \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) R^{-1} \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) \\ &= \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) v_\mu \left(\mathbb{I} \otimes E_{aa}^{(0,k)} \right) = \lambda_a \omega_\mu \otimes E_{aa}^{(0,k)} \end{aligned}$$

for all $a = 0, \dots, k$ and $\mu = 1, \dots, n$. As $v_\mu, R, R^{-1} \in M_{n_0} \otimes \text{DT}_{k+1}$, only "diagonal parts" of R, v_μ, R^{-1} are left when $Rv_\mu R^{-1}$ is sandwiched by $\mathbb{I} \otimes E_{aa}^{(0,k)}$. This corresponds to the first equality. The second equality is due to $R - \mathbb{I}, R^{-1} - \mathbb{I} \in M_{n_0} \otimes \text{DT}_{0, k+1}$. The last equality is because of Condition 5 (iii) for v, ω . (1), (2) of (iv) follows from the fact that $y_{a, \alpha, \beta}^{(l)}$ and $Ry_{a, \alpha, \beta}^{(l)} R^{-1}$ are similar to each other, and v_μ and $Rv_\mu R^{-1}$ are similar to each other with the common invertible operator R . (3) of (iv) follows from the following calculation:

$$\begin{aligned} & Ry_{0, \alpha, \beta}^{(l)} R^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = Ry_{0, \alpha, \beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = R \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) y_{0, \alpha, \beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) \\ &= \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) y_{0, \alpha, \beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) Ry_{0, \alpha, \beta}^{(l)} R^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right). \end{aligned}$$

The first and third equality follows from (41). The second equality is from *Condition 5* (iv)(3) of $y_{0,\alpha,\beta}^{(l)}$. The last equality is from (41) and $R - \mathbb{I} \in M_{n_0} \otimes DT_{0,k+1}$.

Set $Y' := \sum_{i+1 \leq j \leq k: \lambda_j = \lambda_{j-(i+1)}} c_j E_{j,j-(i+1)}^{(0,k)}$. We show (ii), (iii) of *Conditions 6-(i+1)* for $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{Ry_{a,\alpha,\beta}^{(l)} R^{-1}\})$ with respect to $Y + Y'$. (ii) is clear from the definition. To see (iii), note that

$$\begin{aligned} (\mathbb{I} + Y + Y')^l &= (\mathbb{I} + Y)^l + \sum_{k=0}^{l-1} (1 + Y)^k Y' (\mathbb{I} + Y)^{l-(k+1)} + \text{terms with more than one } Y' \\ &= (\mathbb{I} + Y)^l + lY' + \text{elements in } \sum_{a,a': a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}. \end{aligned} \quad (42)$$

We used the fact that terms with more than one Y' belong to $\sum_{a,a': a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$ because $Y' \in \sum_{a,a': a-a' = i+1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$. Similarly, the terms with one Y' and more than zero $Y \in \sum_{a,a': a-a' \geq 1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$ belong to $\sum_{a,a': a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$. By (39) and (42), we have

$$\begin{aligned} \widetilde{Y_{\alpha,\beta}^{(l)}} &:= y_{0,\alpha,\beta}^{(l)} - e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l - \hat{J} \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l \right) + \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l \right) \hat{J} - l e_{\alpha\beta}^{(n_0)} \otimes Y' \Lambda_{\lambda}^l \\ &= y_{0,\alpha,\beta}^{(l)} - e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l - \sum_{j=i+1}^k \left(\lambda_{j-(i+1)}^l J_j e_{\alpha\beta}^{(n_0)} - \lambda_j^l e_{\alpha\beta}^{(n_0)} J_j + \delta_{\lambda_j, \lambda_{j-(i+1)}} c_j l \lambda_j^l e_{\alpha\beta}^{(n_0)} \right) \otimes E_{j,j-(i+1)}^{(0,k)} \\ &\quad - \hat{J} \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l \left((\mathbb{I} + Y)^l - \mathbb{I} \right) \right) + \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l \left((\mathbb{I} + Y)^l - \mathbb{I} \right) \right) \hat{J} \\ &= Y_{\alpha,\beta}^{(l)} - \hat{J} \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l \left((\mathbb{I} + Y)^l - \mathbb{I} \right) \right) + \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l \left((\mathbb{I} + Y)^l - \mathbb{I} \right) \right) \hat{J} \\ &\in M_{n_0} \otimes \sum_{a,a': a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}, \end{aligned}$$

for $\alpha, \beta = 1, \dots, n_0$ and $l \geq l_0$. The last inclusion is because of $\hat{J} \in M_{n_0} \otimes \sum_{a,a': a-a' = i+1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$ and $Y \in \sum_{a,a': a-a' \geq 1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$. Recall that $R := \mathbb{I} - \hat{J}$ and $R^{-1} = \mathbb{I} + \hat{J} + \dots + \hat{J}^k$. We have

$$\begin{aligned} Ry_{0,\alpha,\beta}^{(l)} R^{-1} - e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y + Y')^l &= (\mathbb{I} - \hat{J}) \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l + \hat{J} \left(e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l \right) - e_{\alpha\beta}^{(n_0)} \otimes \left(\Lambda_{\lambda}^l (\mathbb{I} + Y)^l \right) \hat{J} + l e_{\alpha\beta}^{(n_0)} \otimes Y' \Lambda_{\lambda}^l \right) (\mathbb{I} + \hat{J} + \dots + \hat{J}^k) \\ &\quad + (\mathbb{I} - \hat{J}) \widetilde{Y_{\alpha,\beta}^{(l)}} (\mathbb{I} + \hat{J} + \dots + \hat{J}^k) - e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y + Y')^l \\ &= \text{terms with more than one } \hat{J} + \text{terms with more than zero } \hat{J} \text{ and one } Y' \\ &\quad + (\mathbb{I} - \hat{J}) \widetilde{Y_{\alpha,\beta}^{(l)}} (\mathbb{I} + \hat{J}) - e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l \left((\mathbb{I} + Y + Y')^l - (\mathbb{I} + Y)^l - lY' \right), \end{aligned} \quad (43)$$

for $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$.

Note that $\hat{J} \in M_{n_0} \otimes \sum_{a,a': a-a' = i+1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$, $Y \in \sum_{a,a': a-a' \geq 1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$, $Y' \in \sum_{a,a': a-a' = i+1} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$, and $\widetilde{Y_{\alpha,\beta}^{(l)}} \in M_{n_0} \otimes \sum_{a,a': a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$. This observation and (42) implies that the right hand side of (43) is in $M_{n_0} \otimes \sum_{a,a': a-a' \geq i+2} E_{aa}^{(0,k)} M_{k+1} E_{a'a'}^{(0,k)}$. This proves (iii). \square

Lemma 6.7. Let $(n_0, k, \omega, \mathbf{v}, \boldsymbol{\lambda}, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ be a septuplet satisfying Condition 6-0. Suppose that there exist strictly positive operators h_a in M_{n_0} , $a = 0, \dots, k$ with $h_0 = \mathbb{I}$ such that

$$y_{a,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = h_a^{\frac{1}{2}} e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k)}, \quad l \geq l_0, \quad a = 0, \dots, k, \quad \alpha, \beta = 1, \dots, n_0.$$

Then there exist $R \in M_{n_0} \otimes M_{k+1}$, $Y \in \text{DT}_{0,k+1}$, and $\{\hat{y}_{a,\alpha,\beta}^{(l)}\}_{a=0,\dots,k,\alpha,\beta=1,\dots,n_0,l \geq l_0}$ satisfying the followings.

1. We have $[\Lambda_{\boldsymbol{\lambda}}, Y] = 0$.
2. For any $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$, we have $\hat{y}_{0,\alpha,\beta}^{(l)} = e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\boldsymbol{\lambda}}^l (\mathbb{I} + Y)^l$.
3. The septuplet $(n_0, k, \omega, R\mathbf{v}R^{-1}, \boldsymbol{\lambda}, l_0, \{\hat{y}_{a,\alpha,\beta}^{(l)}\})$ satisfies Condition 5.
4. For any $a = 0, \dots, k$, $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$, we have

$$\hat{y}_{a,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = e_{\alpha,\beta}^{(n_0)} \otimes E_{a,0}^{(0,k)}.$$

5. If $X \in \mathcal{K}_l(R\mathbf{v}R^{-1})$, $l \geq l_0$, satisfies $X \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = 0$, then $X = 0$.

6. Set $\tilde{\Lambda} := \Lambda_{\boldsymbol{\lambda}} (\mathbb{I} + Y)$ and $\hat{\Lambda} := \mathbb{I} \otimes \tilde{\Lambda}$. Then we have

$$\begin{aligned} \hat{\Lambda}^l \hat{y}_{a,\alpha,\beta}^{(l_1)} &= \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \hat{\Lambda}^l, \\ \hat{y}_{a,\alpha,\beta}^{(l_1)} \hat{\Lambda}^l &= \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^{-l} f_a^{(0,k)} \right\rangle \hat{\Lambda}^l \hat{y}_{a',\alpha,\beta}^{(l_1)}, \end{aligned} \quad (44)$$

for all $l \in \mathbb{N}$, $l_1 \geq l_0$, $a = 1, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$.

7. For all $l \geq l_0$, $a = 1, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$, we have

$$\hat{y}_{a,\alpha,\beta}^{(l)} \in M_{n_0} \otimes \sum_{\substack{a', a''=0,\dots,k \\ \lambda_{a'} = \lambda_{a''} \lambda_a}} E_{a'a'}^{(0,k)} M_{k+1} E_{a''a''}^{(0,k)}.$$

Proof. By Lemma 6.5 and the assumption that Condition 6-0 holds, we obtain invertible matrices $R_i \in M_{n_0} \otimes M_{k+1}$, $i = 1, \dots, k$, and lower triangular matrices $Y_i \in \text{DT}_{0,k+1}$, $i = 1, \dots, k$. They satisfy the following properties.

1. $R_i - \mathbb{I} \in M_{n_0} \otimes \text{DT}_{0,k+1}$, $R_i \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = R_i^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = \mathbb{I} \otimes E_{00}^{(0,k)}$.
2. The septuplet $(n_0, k, \omega, (R_i \cdots R_1) \mathbf{v} (R_i \cdots R_1)^{-1}, \boldsymbol{\lambda}, l_0, \{(R_i \cdots R_1) y_{a,\alpha,\beta}^{(l)} (R_i \cdots R_1)^{-1}\})$ satisfies Condition 6- i with respect to $Y_1 + \cdots + Y_i$.

Set $R := R_k \cdots R_1$ and $Y := Y_1 + \cdots + Y_k$. Then R is invertible, $R - \mathbb{I} \in M_{n_0} \otimes \text{DT}_{0,k+1}$ and $R \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = R^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = \mathbb{I} \otimes E_{00}^{(0,k)}$. The matrix Y belongs to $\text{DT}_{0,k+1}$ and the septuplet $(n_0, k, \omega, R\mathbf{v}R^{-1}, \boldsymbol{\lambda}, l_0, \{Ry_{a,\alpha,\beta}^{(l)}R^{-1}\})$ satisfies Condition 6- k with respect to Y .

Now we would like to change the basis $\{Ry_{a,\alpha,\beta}^{(l)}R^{-1}\}$ so that the new basis satisfies 4 of the current Lemma. We set $\tilde{R} := \sum_{a=0}^k h_a^{\frac{1}{2}} \otimes E_{aa}^{(0,k)}$ and define

$$\hat{y}_{a,\alpha,\beta}^{(l)} = \sum_{i=1}^k \sum_{\alpha'=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, \tilde{R}^{-1} R^{-1} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle Ry_{i,\alpha',\beta}^{(l)} R^{-1}, \quad 1 \leq a \leq k, \quad \alpha, \beta = 1, \dots, n_0, \quad l \geq l_0.$$

Furthermore, we set $\hat{y}_{0,\alpha,\beta}^{(l)} := Ry_{0,\alpha,\beta}^{(l)} R^{-1}$, $\alpha, \beta = 1, \dots, n_0$, $l \geq l_0$.

This defines the change of basis of $\mathcal{K}_l(RvR^{-1})$ for $l \geq l_0$. Note that

$$Ry_{a,\alpha,\beta}^{(l)} R^{-1} = \sum_{i=1}^k \sum_{\alpha'=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, R\tilde{R} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle \hat{y}_{i,\alpha',\beta}^{(l)}, \quad 1 \leq a \leq k, \quad \alpha, \beta = 1, \dots, n_0, \quad l \geq l_0.$$

Let us check the properties 1,2,4,5 of Lemma 6.7 for $R \in M_{n_0} \otimes M_{k+1}$, $Y \in \text{DT}_{0,k+1}$, and $\{\hat{y}_{a,\alpha,\beta}^{(l)}\}_{a=0,\dots,k,\alpha,\beta=1,\dots,n_0,l \geq l_0}$. As the septuplet $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{Ry_{a,\alpha,\beta}^{(l)} R^{-1}\})$ satisfies *Condition 6-k* with respect to Y , 1 of Lemma 6.7 holds. The third condition of *Condition 6-k* implies

$$\hat{y}_{0,\alpha,\beta}^{(l)} = Ry_{0,\alpha,\beta}^{(l)} R^{-1} = e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l, \quad \alpha, \beta = 1, \dots, n_0, \quad l \geq l_0,$$

proving 2 of Lemma 6.7.

Next we prove 4 of Lemma 6.7. For any $1 \leq a \leq k$, $\alpha, \beta = 1, \dots, n_0$, and $l \geq l_0$, we have

$$\begin{aligned} \hat{y}_{a,\alpha,\beta}^{(l)} (\mathbb{I} \otimes E_{00}^{(0,k)}) &= \sum_{i=1}^k \sum_{\alpha'=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, \tilde{R}^{-1} R^{-1} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle Ry_{i,\alpha',\beta}^{(l)} R^{-1} (\mathbb{I} \otimes E_{00}^{(0,k)}) \\ &= \sum_{i=1}^k \sum_{\alpha'=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, \tilde{R}^{-1} R^{-1} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle Ry_{i,\alpha',\beta}^{(l)} (\mathbb{I} \otimes E_{00}^{(0,k)}) \\ &= \sum_{i=1}^k \sum_{\alpha'=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, \tilde{R}^{-1} R^{-1} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle R \left(h_i^{\frac{1}{2}} e_{\alpha'\beta}^{(n_0)} \otimes E_{i0}^{(0,k)} \right) \\ &= \sum_{i=1}^k \sum_{\alpha'=1}^{n_0} \sum_{i'=1}^k \sum_{\alpha''=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, \tilde{R}^{-1} R^{-1} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle \left\langle \chi_{\alpha''}^{(n_0)} \otimes f_{i'}^{(0,k)}, R\tilde{R} \left(\chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)} \right) \right\rangle (e_{\alpha''\beta}^{(n_0)} \otimes E_{i'0}^{(0,k)}) \\ &= \sum_{i=0}^k \sum_{\alpha'=1}^{n_0} \sum_{i'=1}^k \sum_{\alpha''=1}^{n_0} \left\langle \chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)}, \tilde{R}^{-1} R^{-1} \left(\chi_{\alpha}^{(n_0)} \otimes f_a^{(0,k)} \right) \right\rangle \left\langle \chi_{\alpha''}^{(n_0)} \otimes f_{i'}^{(0,k)}, R\tilde{R} \left(\chi_{\alpha'}^{(n_0)} \otimes f_i^{(0,k)} \right) \right\rangle (e_{\alpha''\beta}^{(n_0)} \otimes E_{i'0}^{(0,k)}) \\ &= e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k)}. \end{aligned}$$

The second equality is from $R^{-1} (\mathbb{I} \otimes E_{00}^{(0,k)}) = \mathbb{I} \otimes E_{00}^{(0,k)}$, while the third one is by the assumption.

For the fourth and fifth equality, we used the fact that $R\tilde{R}, \tilde{R}^{-1} R^{-1} \in M_{n_0} \otimes \text{DT}_{k+1}$. We also have

$$\hat{y}_{0,\alpha,\beta}^{(l)} (\mathbb{I} \otimes E_{00}^{(0,k)}) = e_{\alpha\beta}^{(n_0)} \otimes \Lambda_{\lambda}^l (\mathbb{I} + Y)^l E_{00}^{(0,k)} = e_{\alpha\beta}^{(n_0)} \otimes E_{00}^{(0,k)}, \quad \alpha, \beta = 1, \dots, n_0, \quad l \geq l_0, \quad (45)$$

because $Y \in \text{DT}_{0,k+1}$ and $\lambda_0 = 1$.

The proof of 5 is the same as the proof of Lemma 5.5 (4), using 4 and the fact that $\{\hat{y}_{a,\alpha,\beta}^{(l)}\}$ is a basis of $\mathcal{K}_l(RvR^{-1})$.

Now we prove 3, of Lemma 6.7, i.e., that the septuplet $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{\hat{y}_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 5*. As the septuplet $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{Ry_{a,\alpha,\beta}^{(l)} R^{-1}\})$ satisfies *Condition 5*, (i), (ii), (iii) of *Condition 5* for $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{\hat{y}_{a,\alpha,\beta}^{(l)}\})$ hold. (1) of (iv) is already shown, and (3) of (iv) is clear from (45). The proof of (iv)(2) is the same as the proof of Lemma 5.5 (2), using 4, 5.

Next we prove the first line of (44) for $l \geq l_0$. We extend this to all $l \in \mathbb{N}$ after that. First, note that $\hat{\Lambda}^l = \sum_{\alpha=1}^{n_0} \hat{y}_{0,\alpha,\alpha}^{(l)} \in \mathcal{K}_l(RvR^{-1})$ for any $l \geq l_0$. Therefore, for any $l, l_1 \geq l_0$, $a = 1, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$, we have

$$\hat{\Lambda}^l \hat{y}_{a,\alpha,\beta}^{(l_1)} - \sum_{\alpha'=1}^k \left\langle f_{\alpha'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle \hat{y}_{\alpha',\alpha,\beta}^{(l_1)} \hat{\Lambda}^l \in \mathcal{K}_{l+l_1}(RvR^{-1}).$$

On the other hand, we have

$$\begin{aligned} & \left(\hat{\Lambda}^l \hat{y}_{a,\alpha,\beta}^{(l_1)} - \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \hat{\Lambda}^l \right) \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) \\ &= \left(\hat{\Lambda}^l \right) \left(e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k)} \right) - \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle \left(e_{\alpha\beta}^{(n_0)} \otimes E_{a'0}^{(0,k)} \right) \right) = 0. \end{aligned}$$

Therefore, from 5, we obtain the first line of (44) with $l \geq l_0$.

Lastly, we prove 6, 7. As we have just proven, the first line of (44) holds for $l \geq l_0$. Therefore, for all $l \geq \max\{l_0, 2k\}$, $l_1 \geq l_0$, $a = 1, \dots, k$, $b, b' = 0, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$, we have

$$\begin{aligned} & \sum_{j=0}^k {}_l C_j \cdot \lambda_b^l \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\mathbb{I} \otimes Y^j \right) \hat{y}_{a,\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) = \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \hat{\Lambda}^l \hat{y}_{a,\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \hat{\Lambda}^l \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \lambda_a^l \lambda_{b'}^l \sum_{j_1=0}^k {}_l C_{j_1} \sum_{j_2=0}^k {}_l C_{j_2} \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes Y^{j_2} \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \lambda_a^l \lambda_{b'}^l \sum_{j_1=0}^k \sum_{j_2=0}^k \sum_{i=0}^{j_1+j_2} \alpha_{(j_1,j_2)}^{(i)} \cdot {}_l C_i \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes Y^{j_2} \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right). \end{aligned}$$

In the last line we used Lemma E.1. Let $\{s_i\}_{i=1}^{m_1}$ be the set of distinct elements in $\{\lambda_j\}_{j=0}^k \cup \{\lambda_j \lambda_{j'}\}_{j,j'=1}^k$. Applying Lemma C.7 of Part I, with (l, k, m) replaced by $(\max\{l_0, 2k+1\}, 2k+1, m_1)$, and $\{s_i\}_{i=1}^{m_1}$, we obtain

$$\begin{aligned} & \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\mathbb{I} \otimes Y^j \right) \hat{y}_{a,\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \begin{cases} \sum_{\substack{j_1, j_2=0, \dots, k: \\ j \leq j_1+j_2}} \alpha_{(j_1,j_2)}^{(j)} \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes Y^{j_2} \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) & \text{if } \lambda_b = \lambda_a \lambda_{b'}, \\ 0 & \text{if } \lambda_b \neq \lambda_a \lambda_{b'}, \end{cases} \end{aligned}$$

for $j = 0, \dots, k$, $l_1 \geq l_0$, $a = 1, \dots, k$, $b, b' = 0, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$. The case with $j = 0$ proves 7.

Similarly, we have

$$0 = \sum_{\substack{j_1, j_2=0, \dots, k: \\ j \leq j_1+j_2}} \alpha_{(j_1,j_2)}^{(j)} \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes Y^{j_2} \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right),$$

for $k < j \leq 2k$, if $\lambda_b = \lambda_a \lambda_{b'}$, $a = 1, \dots, k$, $b, b' = 0, \dots, k$, $l_1 \geq l_0$, and $\alpha, \beta = 1, \dots, n_0$.

Let us consider the case with $\lambda_b = \lambda_a \lambda_{b'}$. We claim

$$\begin{aligned} & \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I}_{M_{k+1}} + (\mathbb{I}_{M_{k+1}} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \begin{cases} \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\mathbb{I} \otimes Y^j \right) \hat{y}_{a,\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) & \text{if } 0 \leq j \leq k \\ 0 & \text{if } k < j \leq 2k \end{cases}, \end{aligned} \quad (46)$$

for $a = 1, \dots, k$, $b, b' = 0, \dots, k$, $l_1 \geq l_0$, and $\alpha, \beta = 1, \dots, n_0$. Here, $Y \otimes \mathbb{I}_{M_{k+1}} + (\mathbb{I}_{M_{k+1}} + Y) \otimes Y \in M_{k+1} \otimes M_{k+1}$, and $\left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I}_{M_{k+1}} + (\mathbb{I}_{M_{k+1}} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle$ denotes a matrix in M_{k+1} such that

$$\left\langle \xi, \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I}_{M_{k+1}} + (\mathbb{I}_{M_{k+1}} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \eta \right\rangle = \left\langle \left(f_{a'}^{(0,k)} \otimes \xi \right), (Y \otimes \mathbb{I}_{M_{k+1}} + (\mathbb{I}_{M_{k+1}} + Y) \otimes Y)^j \left(f_a^{(0,k)} \otimes \eta \right) \right\rangle,$$

for $\xi, \eta \in \mathbb{C}^{k+1}$. From Lemma E.1, we have

$$\begin{aligned} & \sum_{\substack{j_1, j_2=0, \dots, k: \\ j \leq j_1+j_2}} \alpha_{(j_1, j_2)}^{(j)} \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \sum_{\substack{j_1, j_2=0, \dots, k: \\ j_2 \leq j \leq j_1+j_2}} j C_{j_2} \cdot \sum_{i=0}^{j_2} \delta_{j_1, j-i} \cdot j_2 C_i \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right). \end{aligned} \quad (47)$$

If $0 \leq j \leq k$, we have

$$\begin{aligned} (47) &= \sum_{j_2=0}^j \sum_{j_1=j-j_2}^j j C_{j_2} \cdot j_2 C_{j-j_1} \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \sum_{j_2=0}^j \sum_{i=0}^{j_2} j C_{j_2} \cdot j_2 C_i \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j-i} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \sum_{j_2=0}^j \sum_{i=0}^{j_2} j C_{j_2} \cdot j_2 C_i \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j-j_2} Y^{j_2-i} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \sum_{j_2=0}^j j C_{j_2} \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j-j_2} (\mathbb{I} + Y)^{j_2} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \hat{y}_{a', \alpha, \beta}^{(l_1)} \left(\mathbb{I} \otimes \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I} + (\mathbb{I} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right). \end{aligned} \quad (48)$$

If $k < j \leq 2k$, then

$$\begin{aligned} (47) &= \sum_{j_2=0}^k \sum_{j_1=j-j_2}^k j C_{j_2} \cdot j_2 C_{j-j_1} \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \sum_{j_2=0}^j \sum_{j_1=j-j_2}^j j C_{j_2} \cdot j_2 C_{j-j_1} \cdot \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, Y^{j_1} f_a^{(0,k)} \right\rangle \hat{y}_{a', \alpha, \beta}^{(l_1)} (\mathbb{I} \otimes Y^{j_2}) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right) \\ &= \left(\mathbb{I} \otimes E_{bb}^{(0,k)} \right) \left(\sum_{a'=1}^k \hat{y}_{a', \alpha, \beta}^{(l_1)} \left(\mathbb{I} \otimes \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I} + (\mathbb{I} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \right) \right) \left(\mathbb{I} \otimes E_{b'b'}^{(0,k)} \right). \end{aligned} \quad (49)$$

In the second equality, we used $Y^{j_1} = 0$ for $j_1 > k$. Hence we proved the claim (46).

By (46) and 7, we obtain

$$\begin{aligned}
\hat{\Lambda}^l \hat{y}_{a,\alpha,\beta}^{(l_1)} &= \sum_{j=0}^{\min\{k,l\}} {}_l C_j \cdot (\mathbb{I} \otimes \Lambda_{\lambda}^l) (\mathbb{I} \otimes Y^j) \hat{y}_{a,\alpha,\beta}^{(l_1)} \\
&= \sum_{j=0}^{\min\{k,l\}} {}_l C_j \cdot (\mathbb{I} \otimes \Lambda_{\lambda}^l) \left(\sum_{a'=1}^k \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I} + (\mathbb{I} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \right) \right) \\
&= \sum_{j=0}^{\min\{k,l\}} {}_l C_j \cdot \lambda_a^l \left(\sum_{a'=1}^k \hat{y}_{a',\alpha,\beta}^{(l_1)} (\mathbb{I} \otimes \Lambda_{\lambda}^l) \left(\mathbb{I} \otimes \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I} + (\mathbb{I} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \right) \right) \\
&= \sum_{j=0}^l {}_l C_j \cdot \lambda_a^l \left(\sum_{a'=1}^k \hat{y}_{a',\alpha,\beta}^{(l_1)} (\mathbb{I} \otimes \Lambda_{\lambda}^l) \left(\mathbb{I} \otimes \left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I} + (\mathbb{I} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle \right) \right) \\
&= \lambda_a^l \left(\sum_{a'=1}^k \hat{y}_{a',\alpha,\beta}^{(l_1)} \left(\mathbb{I} \otimes \Lambda_{\lambda}^l \left\langle f_{a'}^{(0,k)}, ((\mathbb{I} + Y) \otimes (\mathbb{I} + Y))^l f_a^{(0,k)} \right\rangle \right) \right) \\
&= \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha,\beta}^{(l_1)} \hat{\Lambda}^l,
\end{aligned}$$

for all $l \in \mathbb{N}$, $l_1 \geq l_0$, $a = 1, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$. In the second and third equality we used 7. and the fact that $\left\langle f_{a'}^{(0,k)}, (Y \otimes \mathbb{I} + (\mathbb{I} + Y) \otimes Y)^j f_a^{(0,k)} \right\rangle = 0$ unless $\lambda_{a'} = \lambda_a$. For the forth equality we used (46) for $k < j \leq 2k$ and $Y^{k+1} = 0$. This proves the first line of 6. The second line can be checked by substituting the first line to the right hand side of the second equality. \square

Lemma 6.8. *Let $(n_0, k, \omega, \mathbf{v}, \lambda, l_0, \{y_{a,\alpha,\beta}^{(l)}\})$ be a septuplet satisfying Condition 6-0. Suppose that there exist strictly positive elements $h_a \in \mathbb{M}_{n_0}$, $a = 0, \dots, 1$ with $h_0 = \mathbb{I}$ such that*

$$y_{a,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) = h_a^{\frac{1}{2}} e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k)}, \quad l \geq l_0, \quad a = 0, \dots, k, \quad \alpha, \beta = 1, \dots, n_0.$$

Then there exist $R \in \mathbb{M}_{n_0} \otimes \mathbb{M}_{k+1}$, $Y \in \text{DT}_{0,k+1}$, and $D_a \in \text{DT}_{0,k+1}$, $a = 1, \dots, k$, satisfying the followings.

- i We have $[\Lambda_{\lambda}, Y] = 0$.
- ii For any $a = 1, \dots, k$, we have $\Lambda_{\lambda} D_a = \lambda_a D_a \Lambda_{\lambda}$.
- iii For any $a = 1, \dots, k$, we have $D_a E_{00}^{(0,k)} = E_{a0}^{(0,k)}$.
- iv The set of matrices $\{D_a\}_{a=1}^k \cup \{1\}$ is linearly independent.
- v For any $a, a' = 1, \dots, k$, $D_a D_{a'}$ belongs to the linear span of $\{D_b \mid \lambda_a \lambda_{a'} = \lambda_b\}$.
- vi Set $\tilde{\Lambda} := \Lambda_{\lambda} (\mathbb{I} + Y)$. Then we have

$$\begin{aligned}
\tilde{\Lambda}^l D_a &= \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle D_{a'} \tilde{\Lambda}^l \\
D_a \tilde{\Lambda}^l &= \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^{-l} f_a^{(0,k)} \right\rangle \tilde{\Lambda}^l D_{a'}
\end{aligned} \tag{50}$$

for all $l \in \mathbb{N}$ and $a = 1, \dots, k$.

vii For any $l \geq l_0$,

$$\left\{ e_{\alpha\beta}^{(n_0)} \otimes \tilde{\Lambda}^l \mid \alpha, \beta = 1, \dots, n_0 \right\} \cup \left\{ e_{\alpha\beta}^{(n_0)} \otimes D_a \tilde{\Lambda}^l \mid \alpha, \beta = 1, \dots, n_0, a = 1, \dots, k \right\}$$

is a basis of $\mathcal{K}_l(RvR^{-1})$.

viii For each $\mu = 1, \dots, n$, there exist unique $x_{\mu a} \in M_{n_0}$, $a = 1, \dots, k$ such that

$$Rv_\mu R^{-1} = \omega_\mu \otimes \tilde{\Lambda} + \sum_{a=1}^k x_{\mu a} \otimes D_a \tilde{\Lambda}.$$

Proof. Let $R \in M_{n_0} \otimes M_{k+1}$, $Y \in \text{DT}_{0,k+1}$, and $\{\hat{y}_{a,\alpha,\beta}^{(l)}\}_{a=0,\dots,k,\alpha,\beta=1,\dots,n_0,l \geq l_0}$ given in Lemma 6.7. We claim that there exist $D_a \in M_{k+1}$, $a = 1, \dots, k$, such that

$$\hat{y}_{a,\alpha,\beta}^{(l)} = e_{\alpha,\beta}^{(n_0)} \otimes D_a \tilde{\Lambda}^l, \quad \alpha, \beta = 1, \dots, n_0, a = 1, \dots, k, l \geq l_0. \quad (51)$$

Set $\tilde{y}_{a,\alpha,\beta}^{(l)} := \hat{y}_{a,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes \tilde{\Lambda}^{-l} \right)$, and decompose it as $\tilde{y}_{a,\alpha,\beta}^{(l)} = \sum_{ij=0}^k x_{ij}^{(a,\alpha,\beta,l)} \otimes E_{ij}^{(0,k)}$ with $x_{ij}^{(a,\alpha,\beta,l)} \in M_{n_0}$. Note that

$$\begin{aligned} \hat{y}_{0,\alpha_1,\beta_1}^{(l_1)} \hat{y}_{a,\alpha,\beta}^{(l)} \hat{y}_{0,\alpha_2,\beta_2}^{(l_2)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) &= \left(e_{\alpha_1\beta_1}^{(n_0)} \otimes \tilde{\Lambda}^{l_1} \right) \left(e_{\alpha,\beta}^{(n_0)} e_{\alpha_2\beta_2}^{(n_0)} \otimes E_{a,0}^{(0,k)} \right) \\ &= \delta_{\beta,\alpha_2} \delta_{\beta_1,\alpha} \left(e_{\alpha_1\beta_2}^{(n_0)} \otimes \tilde{\Lambda}^{l_1} E_{a,0}^{(0,k)} \right) = \delta_{\beta,\alpha_2} \delta_{\beta_1,\alpha} \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^{l_1} f_a^{(0,k)} \right\rangle e_{\alpha_1\beta_2}^{(n_0)} \otimes E_{a',0}^{(0,k)} \\ &= \delta_{\beta,\alpha_2} \delta_{\beta_1,\alpha} \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^{l_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha_1,\beta_2}^{(l_1+l_2+l)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right), \end{aligned}$$

for any $l, l_1, l_2 \geq l_0$, $\alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2 = 1, \dots, n_0$, $a = 1, \dots, k$. As $\hat{y}_{0,\alpha_1,\beta_1}^{(l_1)} \hat{y}_{a,\alpha,\beta}^{(l)} \hat{y}_{0,\alpha_2,\beta_2}^{(l_2)} - \delta_{\beta,\alpha_2} \delta_{\beta_1,\alpha} \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^{l_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha_1,\beta_2}^{(l_1+l_2+l)}$ belongs to $\mathcal{K}_{l+l_1+l_2}(RvR^{-1})$, this equality and 5 of Lemma 6.7 implies

$$\hat{y}_{0,\alpha_1,\beta_1}^{(l_1)} \hat{y}_{a,\alpha,\beta}^{(l)} \hat{y}_{0,\alpha_2,\beta_2}^{(l_2)} = \delta_{\beta,\alpha_2} \delta_{\beta_1,\alpha} \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^{l_1} f_a^{(0,k)} \right\rangle \hat{y}_{a',\alpha_1,\beta_2}^{(l_1+l_2+l)}, \quad (52)$$

for any $l, l_1, l_2 \geq l_0$, $\alpha, \beta, \alpha_1, \beta_1, \alpha_2, \beta_2 = 1, \dots, n_0$, $a = 1, \dots, k$. The left hand side can be written

$$\begin{aligned} \hat{y}_{0,\alpha_1,\beta_1}^{(l_1)} \hat{y}_{a,\alpha,\beta}^{(l)} \hat{y}_{0,\alpha_2,\beta_2}^{(l_2)} &= \left(e_{\alpha_1\beta_1}^{(n_0)} \otimes \tilde{\Lambda}^{l_1} \right) \tilde{y}_{a,\alpha,\beta}^{(l)} \left(\mathbb{I} \otimes \tilde{\Lambda}^{l_2} \right) \left(e_{\alpha_2\beta_2}^{(n_0)} \otimes \tilde{\Lambda}^{l_2} \right) \\ &= \sum_{ij=0}^k e_{\alpha_1\beta_1}^{(n_0)} x_{ij}^{(a,\alpha,\beta,l)} e_{\alpha_2\beta_2}^{(n_0)} \otimes \tilde{\Lambda}^{l_1} E_{ij}^{(0,k)} \tilde{\Lambda}^{l_2}. \end{aligned} \quad (53)$$

Let $\beta \neq \alpha_2$ or $\beta_1 \neq \alpha$ in (52). Comparing with (53), we obtain

$$\left\langle \chi_{\beta_1}^{(n_0)}, x_{ij}^{(a,\alpha,\beta,l)} \chi_{\alpha_2}^{(n_0)} \right\rangle = 0,$$

for any $i, j = 0, \dots, k$, $l \geq l_0$, $a = 1, \dots, k$, with $\alpha, \beta, \beta_1, \alpha_2 = 1, \dots, n_0$, such that $\beta \neq \alpha_2$ or $\beta_1 \neq \alpha$. This means that there exist $c_{ij}^{(a,\alpha,\beta,l)} \in \mathbb{C}$, $i, j = 0, \dots, k$, $l \geq l_0$, $a = 1, \dots, k$, $\alpha, \beta = 1, \dots, n_0$, such that

$$x_{ij}^{(a,\alpha,\beta,l)} = c_{ij}^{(a,\alpha,\beta,l)} e_{\alpha\beta}^{(n_0)}.$$

Set $Z_{a,\alpha,\beta}^{(l)} := \sum_{ij=0}^k c_{ij}^{(a,\alpha,\beta,l)} E_{ij}^{(0,k)}$. Then we have

$$\hat{y}_{a,\alpha,\beta}^{(l)} = e_{\alpha,\beta}^{(n_0)} \otimes Z_{a,\alpha,\beta}^{(l)}, \quad \alpha, \beta = 1, \dots, n_0, \quad a = 1, \dots, k, \quad l \geq l_0.$$

Furthermore, considering the case $\beta = \alpha_2$ and $\beta_1 = \alpha$ in (52), we see that $\hat{y}_{0,\alpha_1,\alpha}^{(l_1)} \hat{y}_{a,\alpha,\beta}^{(l)} \hat{y}_{0,\beta,\beta_2}^{(l_2)}$ is independent of α, β . Therefore, $Z_{a,\alpha,\beta}^{(l)}$ is α, β -independent. We denote this α, β -independent matrix by $\tilde{Z}_a^{(l)}$. Lastly, we would like to show that $\tilde{Z}_a^{(l)}$ is l -independent. Note that for any $\alpha, \beta = 1, \dots, n_0$, $a = 1, \dots, k$, and $l_1, l_2 \geq l_0$, we have

$$\hat{y}_{a,\alpha,\beta}^{(l_1)} (\mathbb{I} \otimes \tilde{\Lambda}^{l_2}) = \hat{y}_{a,\alpha,\beta}^{(l_1)} \sum_{\gamma=1}^{n_0} \hat{y}_{0,\gamma,\gamma}^{(l_2)} = \hat{y}_{a,\alpha,\beta}^{(l_1+l_2)} = \hat{y}_{a,\alpha,\beta}^{(l_2)} \sum_{\gamma=1}^{n_0} \hat{y}_{0,\gamma,\gamma}^{(l_1)} = \hat{y}_{a,\alpha,\beta}^{(l_2)} (\mathbb{I} \otimes \tilde{\Lambda}^{l_1}),$$

by *Condition 5*. Therefore, multiplying $\mathbb{I} \otimes \tilde{\Lambda}^{-(l_1+l_2)}$ from right, we have $\hat{y}_{a,\alpha,\beta}^{(l_1)} = \hat{y}_{a,\alpha,\beta}^{(l_2)}$. This implies that $\tilde{Z}_a^{(l)}$ is l -independent, $a = 1, \dots, k$. We set $D_a := \tilde{Z}_a^{(l)}$, $a = 1, \dots, k$, and obtain the claim (51).

Next we show $D_a \in \text{DT}_{0,k+1}$. As the septuplet $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{\hat{y}_{a,\alpha,\beta}^{(l)}\})$ satisfies *Condition 5*, we have $RvR^{-1} \hat{y}_{a,\alpha,\beta}^{(l)} \in M_{n_0} \otimes \text{DT}_{k+1}$. We also have $\mathbb{I} \otimes \tilde{\Lambda}^l \in M_{n_0} \otimes \text{DT}_{k+1}$ because $Y \in \text{DT}_{0,k+1}$. Therefore, we have $e_{\alpha,\beta}^{(n_0)} \otimes D_a = \hat{y}_{a,\alpha,\beta}^{(l)} (\mathbb{I} \otimes \tilde{\Lambda}^{-l}) \in M_{n_0} \otimes \text{DT}_{k+1}$. Furthermore, for any $i = 0, \dots, k$ and $a = 1, \dots, k$, we have

$$\begin{aligned} e_{\alpha,\beta}^{(n_0)} \otimes E_{ii}^{(0,k)} D_a E_{ii}^{(0,k)} &= (\mathbb{I} \otimes E_{ii}^{(0,k)}) \hat{y}_{a,\alpha,\beta}^{(l)} (\mathbb{I} \otimes E_{ii}^{(0,k)}) (\mathbb{I} \otimes E_{ii}^{(0,k)} \tilde{\Lambda}^{-l} E_{ii}^{(0,k)}) = (\mathbb{I} \otimes E_{i0}^{(0,k)}) \hat{y}_{a,\alpha,\beta}^{(l)} (\mathbb{I} \otimes E_{0i}^{(0,k)}) \\ &= (\mathbb{I} \otimes E_{i0}^{(0,k)}) (e_{\alpha,\beta}^{(n_0)} \otimes E_{a0}^{(0,k)}) (\mathbb{I} \otimes E_{0i}^{(0,k)}) = 0. \end{aligned}$$

Here, we used the fact that as $\hat{y}_{a,\alpha,\beta}^{(l)} \mathbb{I} \otimes \tilde{\Lambda}^{-l} \in M_{n_0} \otimes \text{DT}_{k+1}$. From this fact, only the "diagonal part" of these matrices can contribute when $\hat{y}_{a,\alpha,\beta}^{(l)} (\mathbb{I} \otimes \tilde{\Lambda}^{-l})$ is sandwiched by $\mathbb{I} \otimes E_{ii}^{(0,k)}$. This corresponds to the first equality. The similar consideration and (iii) of *Condition 5* for $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{\hat{y}_{a,\alpha,\beta}^{(l)}\})$ implies the second equality. For the third equality, we used 4 of Lemma 6.7. This proves $D_a \in \text{DT}_{0,k+1}$.

Now let us check the properties i-vii. i. is from Lemma 6.7. ii. follows from the definition of D_a , and 1, 7 of Lemma 6.7. Note that $Y E_{00}^{(0,k)} = 0$ because $Y \in \text{DT}_{0,k+1}$, $[Y, \Lambda_\lambda] = 0$, and $\lambda_i \neq 1$ if $i \neq 0$. From this and 4 of Lemma 6.7 and (51), we obtain iii. iii implies iv. By 6 of Lemma 6.7 and (51), we have

$$e_{\alpha,\beta}^{(n_0)} \otimes \tilde{\Lambda}^l D_a \tilde{\Lambda}^{l_1} = \sum_{a'=1}^k \left\langle f_{a'}^{(0,k)}, \tilde{\Lambda}^l f_a^{(0,k)} \right\rangle e_{\alpha,\beta}^{(n_0)} \otimes D_{a'} \tilde{\Lambda}^{l_1+l},$$

for all $l \in \mathbb{N}$, $l_1 \geq l_0$, $a = 1, \dots, k$, and $\alpha, \beta = 1, \dots, n_0$. This implies the first equation of vi. The second one follows from this. From 6 of Lemma 6.7 and (51), we have

$$\begin{aligned} \mathbb{I} \otimes D_a D_{a'} &= \sum_{\alpha=1}^{n_0} \hat{y}_{a,\alpha,\alpha}^{(l)} (\mathbb{I} \otimes \tilde{\Lambda}^{-l}) \hat{y}_{a',\alpha,\alpha}^{(l)} (\mathbb{I} \otimes \tilde{\Lambda}^{-l}) = \sum_{\alpha=1}^{n_0} \sum_{b=1}^k \left\langle f_b^{(0,k)}, \tilde{\Lambda}^{-l} f_{a'}^{(0,k)} \right\rangle \hat{y}_{a,\alpha,\alpha}^{(l)} \hat{y}_{b,\alpha,\alpha}^{(l)} (\mathbb{I} \otimes \tilde{\Lambda}^{-2l}) \\ &\in \mathcal{K}_{2l} (RvR^{-1}) (\mathbb{I} \otimes \tilde{\Lambda}^{-2l}) = \text{span} \left\{ \hat{y}_{b,\alpha,\beta}^{(2l)} (\mathbb{I} \otimes \tilde{\Lambda}^{-2l}) \right\}_{b=0,\dots,k, \alpha,\beta=1,\dots,n_0} = M_{n_0} \otimes \text{span} (\{\mathbb{I}\} \cup \{D_b\}_{b=1}^k), \end{aligned}$$

for all $a, a' = 1, \dots, k$, $l \geq l_0$.

On the other hand, from ii, we have $\Lambda_\lambda D_a D_{a'} = \lambda_a \lambda_{a'} D_a D_{a'} \Lambda_\lambda$. Combining these and iv, we obtain v. vii follows directly from *Condition 5* (iv)(1) of $(n_0, k, \omega, RvR^{-1}, \lambda, l_0, \{\hat{y}_{a,\alpha,\beta}^{(l)}\})$.

Finally, we prove viii. Set $\hat{\Lambda} := \mathbb{I} \otimes \tilde{\Lambda}$. Note that $\hat{\Lambda}^l = \mathbb{I} \otimes \tilde{\Lambda}^l = \sum_{\gamma=1}^{n_0} \hat{y}_{0\gamma\gamma}^{(l)} \in \mathcal{K}_l(RvR^{-1})$ for $l \geq l_0$. Let $\mu = 1, \dots, n$, and $l_2 \geq l_0$. Then we have

$$\begin{aligned}
Rv_\mu R^{-1} \hat{\Lambda}^{l_2} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) &= Rv_\mu R^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) \\
&= \sum_{a=1}^k \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)} \otimes f_a^{(0,k)}, Rv_\mu R^{-1} \left(\chi_\beta^{(n_0)} \otimes f_0^{(0,k)} \right) \right\rangle \left(e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k)} \right) \\
&\quad + \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) Rv_\mu R^{-1} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) \\
&= \sum_{a=1}^k \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)} \otimes f_a^{(0,k)}, Rv_\mu R^{-1} \left(\chi_\beta^{(n_0)} \otimes f_0^{(0,k)} \right) \right\rangle \left(e_{\alpha\beta}^{(n_0)} \otimes E_{a0}^{(0,k)} \right) \\
&\quad + \omega_\mu \otimes E_{00}^{(0,k)} \\
&= \sum_{a=1}^k \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)} \otimes f_a^{(0,k)}, Rv_\mu R^{-1} \left(\chi_\beta^{(n_0)} \otimes f_0^{(0,k)} \right) \right\rangle \hat{y}_{a,\alpha,\beta}^{(l_2+1)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right) \\
&\quad + \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)}, \omega_\mu \chi_\beta^{(n_0)} \right\rangle \hat{y}_{0,\alpha,\beta}^{(l_2+1)} \left(\mathbb{I} \otimes E_{00}^{(0,k)} \right)
\end{aligned}$$

As $\hat{y}_{b,\alpha,\beta}^{(l_2+1)}$ and $Rv_\mu R^{-1} \hat{\Lambda}^{l_2}$ belong to $\mathcal{K}_{l_2+1}(Rv_\mu R^{-1})$, from 5 of Lemma 6.7, we obtain

$$\begin{aligned}
Rv_\mu R^{-1} \hat{\Lambda}^{l_2} &= \sum_{a=1}^k \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)} \otimes f_a^{(0,k)}, Rv_\mu R^{-1} \left(\chi_\beta^{(n_0)} \otimes f_0^{(0,k)} \right) \right\rangle \hat{y}_{a,\alpha,\beta}^{(l_2+1)} + \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)}, \omega_\mu \chi_\beta^{(n_0)} \right\rangle \hat{y}_{0,\alpha,\beta}^{(l_2+1)} \\
&= \sum_{a=1}^k \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)} \otimes f_a^{(0,k)}, Rv_\mu R^{-1} \left(\chi_\beta^{(n_0)} \otimes f_0^{(0,k)} \right) \right\rangle e_{\alpha\beta}^{(n_0)} \otimes D_a \tilde{\Lambda}^{l_2+1} + \omega_\mu \otimes \tilde{\Lambda}^{l_2+1}.
\end{aligned}$$

Multiplying $\tilde{\Lambda}^{-l_2}$ from right, we obtain

$$\begin{aligned}
Rv_\mu R^{-1} &= \sum_{a=1}^k \sum_{\alpha, \beta=1}^{n_0} \left\langle \chi_\alpha^{(n_0)} \otimes f_a^{(0,k)}, Rv_\mu R^{-1} \left(\chi_\beta^{(n_0)} \otimes f_0^{(0,k)} \right) \right\rangle e_{\alpha\beta}^{(n_0)} \otimes D_a \tilde{\Lambda} + \omega_\mu \otimes \tilde{\Lambda}.
\end{aligned}$$

This gives the representation of $Rv_\mu R^{-1}$ in viii. The uniqueness of the representation follows from iv. \square

We would like to apply the last Lemma to our setting. In order to do so, we need the following Lemma.

Lemma 6.9. *Let $n_0^{(\sigma)}, n \in \mathbb{N}$ and $\omega^{(\sigma)} \in \text{Prim}_u(n, n_0^{(\sigma)})$, for each $\sigma = L, R$. Let ρ_σ be the faithful $T_{\omega^{(\sigma)}}$ -invariant state. Let $\tilde{\rho}$ be the density matrix of ρ_L . Let φ be a state on $\mathcal{A}_{\mathbb{Z}}$. For each $N \in \mathbb{N}$, let D_N be the density matrix of $\varphi|_{\mathcal{A}_{[0, N-1]}}$, and assume $\sup_N \text{rank } D_N < \infty$. Assume that $(M_{n_0^{(R)}}, \omega^{(R)}, \rho_R)$ right-generates φ and that $(M_{n_0^{(L)}}, \omega^{(L)}, \rho_L)$ left-generates φ . Then there exists an antiunitary $J : \mathbb{C}^{n_0^{(R)}} \rightarrow \mathbb{C}^{n_0^{(L)}}$ and $c \in \mathbb{T}$ such that*

$$\omega_\mu^{(R)} = c J^* \tilde{\rho}^{-\frac{1}{2}} \left(\omega_\mu^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J \quad \mu = 1, \dots, n.$$

Proof. As $\omega^{(L)}$ belongs to $\text{Prim}_u(n, n_0^{(L)})$, the density matrix $\tilde{\rho}$ is strictly positive and has a decomposition $\tilde{\rho} = \sum_{i=1}^{n_0^{(L)}} \lambda_i |\xi_i\rangle \langle \xi_i|$ with a CONS $\{\xi_i\}_i$ of $\mathbb{C}^{n_0^{(L)}}$ and $\lambda_i > 0$. Let J_0 be the operator of complex conjugation with respect to this CONS $\{\xi_i\}_i$. Then J_0 is an antiunitary operator on $\mathbb{C}^{n_0^{(L)}}$ such that $J_0^2 = 1$ and $J_0 = J_0^*$. Furthermore, we have

$$\rho_L(J_0 X^* J_0) = \text{Tr } \tilde{\rho} (J_0 X^* J_0) = \text{Tr } \tilde{\rho} X = \rho_L(X), \quad X \in M_{n_0^{(L)}}. \quad (54)$$

Define $\tilde{\omega}^{(L)} = (\tilde{\omega}_\mu^{(L)})_{\mu=1}^n$ by

$$\tilde{\omega}_\mu^{(L)} := J_0 \tilde{\rho}^{-\frac{1}{2}} \left(\omega_\mu^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J_0, \quad \mu = 1, \dots, n.$$

As $\omega^{(L)}$ belongs to $\text{Prim}_u(n, n_0^{(L)})$, and ρ is $T_{\omega^{(L)}}$ -invariant, $\tilde{\omega}^{(L)}$ also belongs to $\text{Prim}_u(n, n_0^{(L)})$. From (54) and the fact that $T_{\omega^{(L)}}$ is unital, ρ_L is $T_{\tilde{\omega}^{(L)}}$ -invariant. We claim that $(M_{n_0^{(L)}}, \tilde{\omega}^{(L)}, \rho_L)$ right generates φ . This can be seen by

$$\begin{aligned} & \rho_L \left(\tilde{\omega}_{\mu_a}^{(L)} \tilde{\omega}_{\mu_{a+1}}^{(L)} \cdots \tilde{\omega}_{\mu_{a+l-1}}^{(L)} \left(\tilde{\omega}_{\nu_{a+l-1}}^{(L)} \right)^* \cdots \left(\tilde{\omega}_{\nu_{a+1}}^{(L)} \right)^* \left(\tilde{\omega}_{\nu_a}^{(L)} \right)^* \right) \\ &= \rho_L \left(J_0 \tilde{\rho}^{-\frac{1}{2}} \left(\omega_{\mu_a}^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J_0 J_0 \tilde{\rho}^{-\frac{1}{2}} \left(\omega_{\mu_{a+1}}^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J_0 \cdots J_0 \tilde{\rho}^{-\frac{1}{2}} \left(\omega_{\mu_{a+l-1}}^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J_0 J_0 \tilde{\rho}^{\frac{1}{2}} \omega_{\nu_{a+l-1}}^{(L)} \tilde{\rho}^{-\frac{1}{2}} J_0 \cdots J_0 \tilde{\rho}^{\frac{1}{2}} \omega_{\nu_a}^{(L)} \tilde{\rho}^{-\frac{1}{2}} J_0 \right) \\ &= \rho_L \left(J_0 \tilde{\rho}^{-\frac{1}{2}} \left(\omega_{\mu_a}^{(L)} \right)^* \left(\omega_{\mu_{a+1}}^{(L)} \right)^* \cdots \left(\omega_{\mu_{a+l-1}}^{(L)} \right)^* \tilde{\rho} \omega_{\nu_{a+l-1}}^{(L)} \cdots \omega_{\nu_{a+1}}^{(L)} \omega_{\nu_a}^{(L)} \tilde{\rho}^{-\frac{1}{2}} J_0 \right) \\ &= \rho_L \left(\tilde{\rho}^{-\frac{1}{2}} \left(\omega_{\nu_{a+l-1}}^{(L)} \cdots \omega_{\nu_{a+1}}^{(L)} \omega_{\nu_a}^{(L)} \right)^* \tilde{\rho} \omega_{\mu_{a+l-1}}^{(L)} \cdots \omega_{\mu_{a+1}}^{(L)} \omega_{\mu_a}^{(L)} \tilde{\rho}^{-\frac{1}{2}} \right) \\ &= \rho_L \left(\omega_{\mu_{a+l-1}}^{(L)} \cdots \omega_{\mu_{a+1}}^{(L)} \omega_{\mu_a}^{(L)} \left(\omega_{\nu_{a+l-1}}^{(L)} \cdots \omega_{\nu_{a+1}}^{(L)} \omega_{\nu_a}^{(L)} \right)^* \right) = \varphi \left(\bigotimes_{i=a}^{a+l-1} e_{\mu_i \nu_i}^{(n)} \right), \end{aligned}$$

for $a \in \mathbb{Z}$, $l \in \mathbb{N}$, $\mu_i, \nu_i \in \{1, \dots, n\}$. For the third equality, we used (54).

By this observation, we can apply Lemma C.4 to $\omega^{(R)}$ and $\tilde{\omega}^{(L)}$. We then obtain a unitary $V : \mathbb{C}^{n_0^{(R)}} \rightarrow \mathbb{C}^{n_0^{(L)}}$ and $c \in \mathbb{T}$ such that

$$V \omega_\mu^{(R)} = c \tilde{\omega}_\mu^{(L)} V = c J_0 \tilde{\rho}^{-\frac{1}{2}} \left(\omega_\mu^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J_0 V \quad \mu = 1, \dots, n.$$

The operator $J := J_0 V$ is an antiunitary from $\mathbb{C}^{n_0^{(R)}}$ to $\mathbb{C}^{n_0^{(L)}}$ such that

$$\omega_\mu^{(R)} = c J^* \tilde{\rho}^{-\frac{1}{2}} \left(\omega_\mu^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J \quad \mu = 1, \dots, n.$$

□

Lemma 6.10. Assume $[A1], [A3], [A4]$, and $[A5]$. There exist $n_0 \in \mathbb{N}$, $\omega \in \text{Prim}_u(n, n_0)$, $k_L, k_R \in \mathbb{N} \cup \{0\}$, $\hat{\mathbf{v}}^{(L)} \in (M_{n_0} \otimes M_{k_L+1})^{\times n}$, $\hat{\mathbf{v}}^{(R)} \in (M_{n_0} \otimes M_{k_R+1})^{\times n}$, $\hat{\boldsymbol{\lambda}}^{(L)} = (\hat{\lambda}_b^{(L)})_{b=0, \dots, k_L} \in \mathbb{C}^{k_L+1}$, $\hat{\boldsymbol{\lambda}}^{(R)} = (\hat{\lambda}_a^{(R)})_{a=0, \dots, k_R} \in \mathbb{C}^{k_R+1}$, $\hat{Y}_L \in \text{UT}_{0, k_L+1}$, $\hat{Y}_R \in \text{DT}_{0, k_R+1}$, $\hat{D}_b^{(L)} \in \text{UT}_{0, k_L+1}$, $\hat{D}_a^{(R)} \in \text{DT}_{0, k_R+1}$, $b = 1, \dots, k_L$, $a = 1, \dots, k_R$, and $l_0 \in \mathbb{N}$, satisfying the followings.

1. We have $[\Lambda_{\hat{\boldsymbol{\lambda}}^{(\sigma)}}, \hat{Y}_\sigma] = 0$, for $\sigma = L, R$.
2. For each $\sigma = L, R$, $\hat{\lambda}_0^{(\sigma)} = 1$ and $0 < |\hat{\lambda}_a^{(\sigma)}| < 1$ for all $a \geq 1$.

3. There exist $\{\hat{x}_{\mu,b}^{(L)}\}_{\mu=1,\dots,n,b=1,\dots,k_L} \subset M_{n_0}$ and $\{\hat{x}_{\mu,a}^{(R)}\}_{\mu=1,\dots,n,a=1,\dots,k_R} \subset M_{n_0}$ such that

$$\begin{aligned}\hat{v}_\mu^{(L)} &= \omega_\mu \otimes \Lambda_{\hat{\lambda}^{(L)}} \left(\mathbb{I} + \hat{Y}_L \right) + \sum_{b=1}^{k_L} \hat{x}_{\mu b}^{(L)} \otimes \Lambda_{\hat{\lambda}^{(L)}} \left(\mathbb{I} + \hat{Y}_L \right) \hat{D}_b^{(L)}, \\ \hat{v}_\mu^{(R)} &= \omega_\mu \otimes \Lambda_{\hat{\lambda}^{(R)}} \left(\mathbb{I} + \hat{Y}_R \right) + \sum_{a=1}^{k_R} \hat{x}_{\mu a}^{(R)} \otimes \hat{D}_a^{(R)} \Lambda_{\hat{\lambda}^{(R)}} \left(\mathbb{I} + \hat{Y}_R \right).\end{aligned}$$

4. We have

$$\begin{aligned}\Lambda_{\hat{\lambda}^{(R)}} \hat{D}_a^{(R)} &= \hat{\lambda}_a^{(R)} \hat{D}_a^{(R)} \Lambda_{\hat{\lambda}^{(R)}} \quad a = 1, \dots, k_R, \\ \Lambda_{\hat{\lambda}^{(L)}} \hat{D}_b^{(L)} &= \left(\hat{\lambda}_b^{(L)} \right)^{-1} \hat{D}_b^{(L)} \Lambda_{\hat{\lambda}^{(L)}} \quad b = 1, \dots, k_L.\end{aligned}$$

5. We have

$$\begin{aligned}\hat{D}_a^{(R)} E_{00}^{(0,k_R)} &= E_{a0}^{(0,k_R)}, \quad a = 1, \dots, k_R, \\ E_{00}^{(0,k_L)} \hat{D}_b^{(L)} &= E_{0b}^{(0,k_L)}, \quad b = 1, \dots, k_L.\end{aligned}$$

6. The set of matrices $\{\hat{D}_a^{(\sigma)}\}_{a=1}^{k_\sigma} \cup \{\mathbb{I}_{k_\sigma+1}\}$ is linearly independent for each $\sigma = L, R$.

7. For any $a, a' = 1, \dots, k_\sigma$, $\sigma = L, R$, $\hat{D}_a^{(\sigma)} \hat{D}_{a'}^{(\sigma)}$ belongs to the linear span of $\{\hat{D}_b^{(\sigma)} \mid \hat{\lambda}_a^{(\sigma)} \hat{\lambda}_{a'}^{(\sigma)} = \hat{\lambda}_b^{(\sigma)}\}$.

8. Set $\tilde{\Lambda}_\sigma := \Lambda_{\hat{\lambda}^{(\sigma)}} \left(\mathbb{I} + \hat{Y}_\sigma \right)$, $\sigma = L, R$. Then we have

$$\begin{aligned}\tilde{\Lambda}_R^l \hat{D}_a^{(R)} &= \sum_{a'=1}^{k_R} \left\langle f_{a'}^{(0,k_R)}, \tilde{\Lambda}_R^l f_a^{(0,k_R)} \right\rangle \hat{D}_{a'}^{(R)} \tilde{\Lambda}_R^l, \quad \hat{D}_a^{(R)} \tilde{\Lambda}_R^l = \sum_{a'=1}^{k_R} \left\langle f_{a'}^{(0,k_R)}, \tilde{\Lambda}_R^{-l} f_a^{(0,k_R)} \right\rangle \tilde{\Lambda}_R^l \hat{D}_{a'}^{(R)}, \\ \hat{D}_b^{(L)} \tilde{\Lambda}_L^l &= \sum_{b'=1}^{k_L} \left\langle f_b^{(0,k_L)}, \tilde{\Lambda}_L^l f_{b'}^{(0,k_L)} \right\rangle \tilde{\Lambda}_L^l \hat{D}_{b'}^{(L)}, \quad \tilde{\Lambda}_L^l \hat{D}_b^{(L)} = \sum_{b'=1}^{k_L} \left\langle f_b^{(0,k_L)}, \tilde{\Lambda}_L^{-l} f_{b'}^{(0,k_L)} \right\rangle \hat{D}_{b'}^{(L)} \tilde{\Lambda}_L^l,\end{aligned}\tag{55}$$

for all $l \in \mathbb{N}$ and $a = 1, \dots, k_R$, $b = 1, \dots, k_L$.

9. For any $l \geq l_0$,

$$\left\{ e_{\alpha\beta}^{(n_0)} \otimes \tilde{\Lambda}_R^l \mid \alpha, \beta = 1, \dots, n_0 \right\} \cup \left\{ e_{\alpha\beta}^{(n_0)} \otimes \hat{D}_a^{(R)} \tilde{\Lambda}_R^l \mid \alpha, \beta = 1, \dots, n_0, a = 1, \dots, k_R \right\}$$

is a basis of $\mathcal{K}_l \left(\hat{\mathbf{v}}^{(R)} \right)$, and

$$\left\{ e_{\alpha\beta}^{(n_0)} \otimes \tilde{\Lambda}_L^l \mid \alpha, \beta = 1, \dots, n_0 \right\} \cup \left\{ e_{\alpha\beta}^{(n_0)} \otimes \tilde{\Lambda}_L^l \hat{D}_b^{(L)} \mid \alpha, \beta = 1, \dots, n_0, b = 1, \dots, k_L \right\}$$

is a basis of $\mathcal{K}_l \left(\hat{\mathbf{v}}^{(L)} \right)$.

10. For the state ω_σ in $[A4]$ and $l \in \mathbb{N}$, $\tau_{y_\sigma} \left(s(\omega_\sigma|_{\mathcal{A}_{\sigma,l}}) \right)$ is equal to the orthogonal projection onto $\Gamma_{l, \hat{\mathbf{v}}^{(\sigma)}}^{(R)} \left(M_{n_0} \otimes M_{k_\sigma+1} \right)$, where $y_R = 0$ and $y_L = l$.

11. Let ρ_0 be the T_ω -invariant state on M_{n_0} . Then $(M_{n_0}, \omega, \rho_0)$ right-generates ω_∞ .

Proof. Recall the Notation 2.5 and Notation 3.6. Let $\sigma = L, R$. We apply Lemma 6.8 to $\omega^{(\sigma)} \in \text{Prim}_u(n, n_0^{(\sigma)})$, $v^{(\sigma)} \in \left(M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1}\right)^{\times n}$, $\lambda^{(\sigma)} = (\lambda_a^{(\sigma)})_{a=0,\dots,k_\sigma} \in \mathbb{C}^{k_\sigma+1}$, $l_0^{(\sigma)} \in \mathbb{N}$, and $\{y_{a,\alpha,\beta}^{(l,\sigma)}\}_{a=0,\dots,k_\sigma,\alpha,\beta=1,\dots,n_0^{(\sigma)}}$ given by Lemma 6.4. We then obtain $R_\sigma \in M_{n_0^{(\sigma)}} \otimes M_{k_\sigma+1}$, $Y_\sigma \in \text{DT}_{0,k_\sigma+1}$, and $D_a^{(\sigma)} \in \text{DT}_{0,k_\sigma+1}$, $a = 1, \dots, k_\sigma$, satisfying the properties i-viii in Lemma 6.8. Let ρ_σ be the $T_{\omega^{(\sigma)}}$ -invariant state for $\sigma = L, R$.

We set $\omega := \omega^{(R)}$, $n_0 := n_0^{(R)}$, $\hat{v}_\mu^{(R)} := R_R v_\mu^{(R)} R_R^{-1}$, $\mu = 1, \dots, n$, $\hat{\lambda}^{(R)} := \lambda^{(R)}$, $\hat{Y}_R := Y_R$, $\hat{D}_a^{(R)} := D_a^{(R)}$, and $l_0 := \max\{l_0^{(L)}, l_0^{(R)}\}$.

In order to define the left parts, we use Lemma 6.9. As ω_∞ is in $\mathcal{S}_{\mathbb{Z}}(H)$, it satisfies the condition in Lemma 6.9 for φ because of [A1] and Lemma 2.1. Recall that $(M_{n_0}, \omega, \rho_R|_{M_{n_0}})$ right-generates ω_∞ and that $(M_{n_0^{(L)}}, \omega^{(L)}, \rho_L|_{M_{n_0^{(L)}}})$ left-generates ω_∞ . Applying Lemma 6.9, we obtain an antiunitary $J : \mathbb{C}^{n_0} \rightarrow \mathbb{C}^{n_0^{(L)}}$ and $c \in \mathbb{T}$ such that

$$\omega_\mu = c J^* \tilde{\rho}^{-\frac{1}{2}} \left(\omega_\mu^{(L)} \right)^* \tilde{\rho}^{\frac{1}{2}} J \quad \mu = 1, \dots, n,$$

where $\tilde{\rho}$ is a strictly positive element in $M_{n_0^{(L)}}$.

Let $J_L : \mathbb{C}^{k_L+1} \rightarrow \mathbb{C}^{k_L+1}$ be the complex conjugation with respect to the standard basis $\{f_i^{(0,k_L)}\}_{i=0,\dots,k_L}$. We set

$$\hat{v}_\mu^{(L)} := c (J^* \otimes J_L^*) \left(\tilde{\rho}^{-\frac{1}{2}} \otimes \mathbb{I} \right) \left(R_L v_\mu^{(L)} R_L^{-1} \right)^* \left(\tilde{\rho}^{\frac{1}{2}} \otimes \mathbb{I} \right) (J \otimes J_L), \quad \mu = 1, \dots, k_L.$$

Furthermore, we set $\hat{\lambda}^{(L)} := \lambda^{(L)}$, $\hat{D}_a^{(L)} := \left(D_a^{(L)} \right)^t$, $\hat{Y}_L := Y_L^t$. It is straightforward to check that all the conditions in the Lemma are satisfied. \square

We make a further simplification.

Lemma 6.11. *In Lemma 6.10, we may assume $\hat{\lambda}^{(\sigma)}$ to satisfy*

$$\begin{aligned} 1 &= \left| \hat{\lambda}_0^{(L)} \right| > \left| \hat{\lambda}_1^{(L)} \right| \geq \left| \hat{\lambda}_2^{(L)} \right| \cdots \geq \left| \hat{\lambda}_{k_L}^{(L)} \right|, \\ 1 &= \left| \hat{\lambda}_0^{(R)} \right| > \left| \hat{\lambda}_1^{(R)} \right| \geq \left| \hat{\lambda}_2^{(R)} \right| \cdots \geq \left| \hat{\lambda}_{k_R}^{(R)} \right| \end{aligned} \quad (56)$$

Proof. We prove for $\sigma = L$. The proof for $\sigma = R$ is the same. As there is nothing to prove if $k_L = 0$, we may assume $k_L \in \mathbb{N}$. We define an order \preceq on \mathbb{C} , by $\lambda \preceq \zeta$ if and only if $|\lambda| < |\zeta|$ or $|\lambda| = |\zeta|$ and $\lambda = |\lambda|e^{i\theta}$, $\zeta = |\zeta|e^{i\varphi}$, with $0 \leq \theta \leq \varphi < 2\pi$. We write $\lambda \prec \zeta$ if $\lambda \preceq \zeta$ and $\lambda \neq \zeta$. For $\hat{\lambda}^{(L)}$ given in Lemma 6.10, we denote the disjoint elements of $\{\hat{\lambda}_i^{(L)}\}_{i=0}^{k_L}$ by s_t , $t = 0, \dots, m$, $m \in \mathbb{N}$, which is ordered as $s_m \prec \cdots \prec s_2 \prec s_1 \prec s_0 = 1$. For each $t = 0, \dots, m$, we define a set \mathfrak{S}_t by $\mathfrak{S}_t := \left\{ i = 0, \dots, k_L \mid \hat{\lambda}_i^{(L)} = s_t \right\}$, and set $n_t := |\mathfrak{S}_t|$, the number of elements in \mathfrak{S}_t . Furthermore, we label the elements in \mathfrak{S}_t as $i_1^{(t)}, i_2^{(t)}, \dots, i_{n_t}^{(t)}$, with labels ordered so that $i_1^{(t)} < i_2^{(t)} < \cdots < i_{n_t}^{(t)}$. For each $i = 0, \dots, k_L$, there exists a unique pair, (t, j) , $t = 0, \dots, m$ and $j = 1, \dots, n_t$ such that $i = i_j^{(t)}$. We use this label to explain the rearrangement of $0, \dots, k_L$. We permute $0, \dots, k_L$ as follows. We know that 0 corresponds to $(0, 1)$ with respect to the label, and from the beginning, it is located at the first position. First, we move $(1, 1)$ to left one by one up to when it get next to $(0, 1)$. Second, we move $(1, 2)$ to left one by one up to when it gets next to $(1, 1)$. We continue this up to when the first $n_1 + 1$ elements of the sequence become $(0, 0), (1, 1), (1, 2), \dots, (1, n_1)$. When this is completed we move $(2, 1)$, to left one by one up to when it gets next to $(1, n_1)$. Repeating the same procedure for $(2, j)$, $j = 1, \dots, n_2$, the sequence is rearranged so that the first $n_1 + n_2 + 1$ elements of the sequence become

$(0, 0), (1, 1), (1, 2), \dots, (1, n_1), (2, 1), \dots, (2, n_2)$. We repeat this procedure up to $t = m$ and obtain the rearranged sequence $(0, 0), (1, 1), (1, 2), \dots, (1, n_1), (2, 1), \dots, (2, n_2), \dots, (m, 1), \dots, (m, n_m)$. We denote by S_{k_L+1} the permutation group of $\{0, \dots, k_L\}$. Note that the above procedure consists of the sequence of transpositions σ_k , $k = 1, \dots, N$ of the form $\sigma_k = (i_k, i_k + 1) \in S_{k_L+1}$ with $i_k \in \{1, \dots, k_L - 1\}$. Furthermore these transpositions satisfy $\sigma_k(0) = 0$,

$$\hat{\lambda}_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1}(i_k)}^{(L)} \prec \hat{\lambda}_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1}(i_k+1)}^{(L)}, \quad (57)$$

and

$$(\sigma_k \circ \dots \circ \sigma_1) \left(i_j^{(t)} \right) < (\sigma_k \circ \dots \circ \sigma_1) \left(i_{j'}^{(t)} \right), \quad t = 0, \dots, m, \quad 1 \leq j < j' \leq n_t, \quad k = 1, \dots, N. \quad (58)$$

Define $\sigma := \sigma_N \circ \dots \circ \sigma_1$. Then we have

$$\tilde{\lambda}^{(L)} := \left(\tilde{\lambda}_i^{(L)} \right)_{i=0, \dots, k_L} := \left(\hat{\lambda}_{\sigma^{-1}(0)}^{(L)}, \hat{\lambda}_{\sigma^{-1}(1)}^{(L)}, \dots, \hat{\lambda}_{\sigma^{-1}(k_L)}^{(L)} \right) = \left(\hat{\lambda}_0^{(L)}, \hat{\lambda}_{i_1^{(1)}}^{(L)}, \dots, \hat{\lambda}_{i_{n_1}^{(1)}}^{(L)}, \hat{\lambda}_{i_1^{(2)}}^{(L)}, \dots, \hat{\lambda}_{i_{n_2}^{(2)}}^{(L)}, \dots, \hat{\lambda}_{i_1^{(m)}}^{(L)}, \dots, \hat{\lambda}_{i_{n_m}^{(m)}}^{(L)} \right).$$

Note that

$$1 = \left| \tilde{\lambda}_0^{(L)} \right| > \left| \tilde{\lambda}_1^{(L)} \right| \geq \left| \tilde{\lambda}_2^{(L)} \right| \dots \geq \left| \tilde{\lambda}_{k_L}^{(L)} \right|.$$

For each $\sigma' \in S_{k_L+1}$, define $U_{\sigma'} \in M_{k_L+1}$ such that $\left\langle f_i^{(0, k_L)}, U_{\sigma'} f_j^{(0, k_L)} \right\rangle := \delta_{\sigma'(i), j}$, $i, j = 0, \dots, k_L$. It is easy to check

$$\left\langle f_i^{(0, k_L)}, U_{\sigma'}^* X U_{\sigma'} f_j^{(0, k_L)} \right\rangle = \left\langle f_{(\sigma')^{-1}(i)}^{(0, k_L)}, X f_{(\sigma')^{-1}(j)}^{(0, k_L)} \right\rangle, \quad i, j = 0, \dots, k_L, \quad X \in M_{k_L+1}. \quad (59)$$

In particular, $U_{\sigma'}$ is unitary, and $U_{\sigma'}^* \Lambda_{\tilde{\lambda}^{(L)}} U_{\sigma'} = \Lambda_{\tilde{\lambda}^{(L)}}$. Furthermore, we have $U_{\sigma'_2} U_{\sigma'_1} = U_{\sigma'_1 \circ \sigma'_2}$, for any $\sigma'_1, \sigma'_2 \in S_{k_L+1}$.

We set

$$\tilde{Y}_L := U_{\sigma}^* \hat{Y}_L U_{\sigma}, \quad \tilde{D}_b^{(L)} := U_{\sigma}^* \hat{D}_{\sigma^{-1}(b)}^{(L)} U_{\sigma}, \quad \tilde{v}_{\mu}^{(L)} := (\mathbb{I} \otimes U_{\sigma}^*) \hat{v}_{\mu}^{(L)} (\mathbb{I} \otimes U_{\sigma}).$$

It is straightforward to check that $n_0 \in \mathbb{N}$, $\omega \in \text{Prim}_u(n, n_0)$, $k_L \in \mathbb{N} \cup \{0\}$ of Lemma 6.10, $\tilde{\mathbf{v}}^{(L)} \in (M_{n_0} \otimes M_{k_L+1})^{\times n}$, $\tilde{\lambda}^{(L)} = (\tilde{\lambda}_b^{(L)})_{b=0, \dots, k_L} \in \mathbb{C}^{k_L+1}$, $\tilde{Y}_L \in M_{k_L+1}$, $\tilde{D}_b^{(L)} \in M_{k_L+1}$, $b = 1, \dots, k_L$, and $l_0 \in \mathbb{N}$ (given in Lemma 6.10), satisfy the "left part" of 1-11 of Lemma 6.10, (replacing $\hat{\cdot}$ by $\tilde{\cdot}$)

We have to prove that $\tilde{Y}_L, \tilde{D}_b \in \text{UT}_{0, k_L+1}$. To prove $\tilde{Y}_L \in \text{UT}_{0, k_L+1}$, note that

$$\left\langle f_i^{(0, k_L)}, \tilde{Y}_L f_j^{(0, k_L)} \right\rangle = \left\langle f_{(\sigma)^{-1}(i)}^{(0, k_L)}, \hat{Y}_L f_{(\sigma)^{-1}(j)}^{(0, k_L)} \right\rangle, \quad i, j = 0, \dots, k_L.$$

The right hand side is zero if $\hat{\lambda}_{(\sigma)^{-1}(i)}^{(L)} \neq \hat{\lambda}_{(\sigma)^{-1}(j)}^{(L)}$. If $\hat{\lambda}_{(\sigma)^{-1}(i)}^{(L)} = \hat{\lambda}_{(\sigma)^{-1}(j)}^{(L)}$ and $i \geq j$, then by (58), we have $(\sigma)^{-1}(i) \geq (\sigma)^{-1}(j)$. As $\hat{Y}_L \in \text{UT}_{0, k_L+1}$, in this case, the right hand side of the above equation is zero. Therefore, we obtain $\tilde{Y}_L \in \text{UT}_{0, k_L+1}$.

Next, to prove $\tilde{D}_b \in \text{UT}_{0, k_L+1}$, note that

$$\left\langle f_i^{(0, k_L)}, U_{\sigma}^* \hat{D}_b^{(L)} U_{\sigma} f_j^{(0, k_L)} \right\rangle = \left\langle f_{(\sigma)^{-1}(i)}^{(0, k_L)}, \hat{D}_b^{(L)} f_{(\sigma)^{-1}(j)}^{(0, k_L)} \right\rangle, \quad i, j = 0, \dots, k_L.$$

We consider the following proposition for $k = 0, \dots, N$:

$$(P_k): \text{ If } 0 \leq j \leq i \leq k_L, \text{ then } \left\langle f_{(\sigma_k \circ \dots \circ \sigma_1)^{-1}(i)}^{(0, k_L)}, \hat{D}_b^{(L)} f_{(\sigma_k \circ \dots \circ \sigma_1)^{-1}(j)}^{(0, k_L)} \right\rangle = 0.$$

(Regard $\sigma_k \circ \dots \circ \sigma_1$ to be identity for $k = 0$.) It suffices to prove (P_N) . (P_0) is true, for $\hat{D}_b^{(L)} \in \text{UT}_{0, k_L+1}$.

Assume that (P_{k-1}) holds for some $1 \leq k \leq N$. We claim that (P_k) holds. Consider $0 \leq j \leq i \leq k_L$. If $i = j$, then $\left\langle f_{(\sigma_k \circ \dots \circ \sigma_1)^{-1}(i)}^{(0,k_L)}, \hat{D}_b^{(L)} f_{(\sigma_k \circ \dots \circ \sigma_1)^{-1}(i)}^{(0,k_L)} \right\rangle$ is zero because the diagonal elements of $\hat{D}_b^{(L)}$ are zero. Suppose that $j < i$. Note that

$$\left\langle f_{(\sigma_k \circ \dots \circ \sigma_1)^{-1}(i)}^{(0,k_L)}, \hat{D}_b^{(L)} f_{(\sigma_k \circ \dots \circ \sigma_1)^{-1}(j)}^{(0,k_L)} \right\rangle = \left\langle f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1} \circ \sigma_k^{-1}(i)}^{(0,k_L)}, \hat{D}_b^{(L)} f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1} \circ \sigma_k^{-1}(j)}^{(0,k_L)} \right\rangle. \quad (60)$$

Recall that $\sigma_k = (i_k, i_k + 1)$. If $i, j \notin \{i_k, i_k + 1\}$, then $\sigma_k^{-1}(i) = i > j = \sigma_k^{-1}(j)$. If $i \in \{i_k, i_k + 1\}$ and $j \notin \{i_k, i_k + 1\}$ (resp. $j \in \{i_k, i_k + 1\}$ and $i \notin \{i_k, i_k + 1\}$), then

$$\sigma_k^{-1}(i) = i_k \text{ or } i_k + 1 > j = \sigma_k^{-1}(j), \quad (\text{resp. } \sigma_k^{-1}(j) = i_k \text{ or } i_k + 1 < i = \sigma_k^{-1}(i)).$$

Therefore, if $i \notin \{i_k, i_k + 1\}$ or $j \notin \{i_k, i_k + 1\}$, we have $\sigma_k^{-1}(i) > \sigma_k^{-1}(j)$, and the assumption (P_{k-1}) implies the right hand side of (60) to be zero.

Let us consider the case $i, j \in \{i_k, i_k + 1\}$, i.e., $i = i_k + 1$ and $j = i_k$. By (57) and the definition of \prec combined with the fact that $|\hat{\lambda}_b^{(L)}| < 1$, we have

$$\hat{\lambda}_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1}(i_k+1)}^{(L)} \neq \hat{\lambda}_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1}(i_k)}^{(L)} \hat{\lambda}_b^{(L)}.$$

Therefore, in this case, the right hand side of (60) is

$$\begin{aligned} & \left\langle f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1} \circ \sigma_k^{-1}(i)}^{(0,k_L)}, \hat{D}_b^{(L)} f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1} \circ \sigma_k^{-1}(j)}^{(0,k_L)} \right\rangle = \left\langle f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1} \circ \sigma_k^{-1}(i_k+1)}^{(0,k_L)}, \hat{D}_b^{(L)} f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1} \circ \sigma_k^{-1}(i_k)}^{(0,k_L)} \right\rangle \\ & = \left\langle f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1}(i_k)}^{(0,k_L)}, \hat{D}_b^{(L)} f_{(\sigma_{k-1} \circ \dots \circ \sigma_1)^{-1}(i_k+1)}^{(0,k_L)} \right\rangle = 0, \end{aligned}$$

by property 4 of Lemma 6.10. This completes the proof of the induction (P_k) . \square

7 Derivation of $\mathbb{B} \in \text{Class A}$

In this section, we prove the following Lemma.

Lemma 7.1. *Assume $[A1], [A3], [A4]$, and $[A5]$. Then there exists $\mathbb{B} \in \text{Class A}$ with respect to a septuplet $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$, satisfying the following conditions.*

1. For $m' \geq m_{\mathbb{B}}$, we have $\mathcal{S}_{[0,\infty)}(H_{\Phi_{m',\mathbb{B}}}) = \mathcal{S}_{[0,\infty)}(H)$, and $\mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m',\mathbb{B}}}) = \mathcal{S}_{(-\infty,-1]}(H)$.
2. For the state ω_L in $[A4]$, $\pi(s(\omega_L|_{\mathcal{A}_{L,l}}))$ is equal to the orthogonal projection onto $\Gamma_{l,\mathbb{B}}^{(R)} \left(M_{n_0} \otimes P_L^{(k_R,k_L)} M_{k_L+k_R+1} P_L^{(k_R,k_L)} \right)$ for all $l \in \mathbb{N}$.
3. For the state ω_R in $[A4]$, $s(\omega_R|_{\mathcal{A}_{R,l}})$ is equal to the orthogonal projection onto $\Gamma_{l,\mathbb{B}}^{(R)} \left(M_{n_0} \otimes P_R^{(k_R,k_L)} M_{k_L+k_R+1} P_R^{(k_R,k_L)} \right)$, for all $l \in \mathbb{N}$.
4. We have $\omega_\infty = \omega_{\mathbb{B},\infty}$, where $\omega_{\mathbb{B},\infty}$ is given in Lemma 3.16 of Part I.

The n -tuple \mathbb{B} in the above Lemma is defined as follows.

Lemma 7.2. *Assume $[A1], [A3], [A4]$, and $[A5]$. Then there exist $n_0 \in \mathbb{N}$, $k_L, k_R \in \mathbb{N} \cup \{0\}$, $\omega \in \text{Prim}_u(n, n_0)$, $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$, $l_0 \in \mathbb{N}$, and $\{x_{\mu,b}^{(L)}\}_{\mu=1,\dots,n,b=1,\dots,k_L}, \{x_{\mu,a}^{(R)}\}_{\mu=1,\dots,n,a=1,\dots,k_R} \subset M_{n_0}$ satisfying the followings.*

1. Define $\mathbb{B} \in (\mathbb{M}_{n_0} \otimes \mathbb{M}_{k_L+k_R+1})^{\times n}$ by

$$B_\mu = \omega_\mu \otimes \Lambda_\lambda (\mathbb{I} + Y) + \sum_{b=1}^{k_L} x_{\mu b}^{(L)} \otimes \Lambda_\lambda (\mathbb{I} + Y) I_L^{(k_R, k_L)}(G_b) + \sum_{a=1}^{k_R} x_{\mu a}^{(R)} \otimes I_R^{(k_R, k_L)}(D_a) \Lambda_\lambda (\mathbb{I} + Y), \quad (61)$$

for $\mu = 1, \dots, n$. Then

$$\begin{aligned} \mathcal{K}_l(\mathbb{B}) \hat{P}_R^{(n_0, k_R, k_L)} &= \mathbb{M}_{n_0} \otimes \text{span} \left\{ I_R^{(k_R, k_L)}(\mathbb{I}), I_R^{(k_R, k_L)}(D_a), a = 1, \dots, k_R \right\} \Lambda_\lambda^l(\mathbb{I} + Y)^l \\ \hat{P}_L^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B}) &= \mathbb{M}_{n_0} \otimes \Lambda_\lambda^l(\mathbb{I} + Y)^l \text{span} \left\{ I_L^{(k_R, k_L)}(\mathbb{I}), I_L^{(k_R, k_L)}(G_b), b = 1, \dots, k_L \right\}, \end{aligned}$$

for all $l \geq l_0$.

2. For the state ω_L in [A4], $\tau_l(s(\omega_L|_{\mathcal{A}_{L,l}}))$ is equal to the orthogonal projection onto

$$\Gamma_{l, \mathbb{B}}^{(R)} \left(\mathbb{M}_{n_0} \otimes P_L^{(k_R, k_L)} \mathbb{M}_{k_L+k_R+1} P_L^{(k_R, k_L)} \right), \text{ for all } l \in \mathbb{N}.$$

3. For the state ω_R in [A4], $s(\omega_R|_{\mathcal{A}_{R,l}})$ is equal to the orthogonal projection onto

$$\Gamma_{l, \mathbb{B}}^{(R)} \left(\mathbb{M}_{n_0} \otimes P_R^{(k_R, k_L)} \mathbb{M}_{k_L+k_R+1} P_R^{(k_R, k_L)} \right), \text{ for all } l \in \mathbb{N}.$$

4. Let ρ_0 be the T_ω -invariant state on \mathbb{M}_{n_0} . Then $(\mathbb{M}_{n_0}, \omega, \rho_0)$ right-generates ω_∞ .

Proof. Let $n_0 \in \mathbb{N}$, $\omega \in \text{Prim}_u(n, n_0)$, $k_L, k_R \in \mathbb{N} \cup \{0\}$, $\hat{\mathbf{v}}^{(L)} \in (\mathbb{M}_{n_0} \otimes \mathbb{M}_{k_L+1})^{\times n}$, $\hat{\mathbf{v}}^{(R)} \in (\mathbb{M}_{n_0} \otimes \mathbb{M}_{k_R+1})^{\times n}$, $\hat{\lambda}^{(L)} = (\hat{\lambda}_b^{(L)})_{b=0, \dots, k_L} \in \mathbb{C}^{k_L+1}$, $\hat{\lambda}^{(R)} = (\hat{\lambda}_a^{(R)})_{a=0, \dots, k_R} \in \mathbb{C}^{k_R+1}$, $\hat{Y}_L \in \text{UT}_{0, k_L+1}$, $\hat{Y}_R \in \text{DT}_{0, k_R+1}$, $\hat{D}_b^{(L)} \in \text{UT}_{0, k_L+1}$, $\hat{D}_a^{(R)} \in \text{DT}_{0, k_R+1}$, $b = 1, \dots, k_L$, $a = 1, \dots, k_R$, and $l_0 \in \mathbb{N}$, satisfying the properties 1-10 of Lemma 6.10 and (56). We also use $\{\hat{x}_{\mu, b}^{(L)}\}_{\mu=1, \dots, n, b=1, \dots, k_L} \subset \mathbb{M}_{n_0}$ and $\{\hat{x}_{\mu, a}^{(R)}\}_{\mu=1, \dots, n, a=1, \dots, k_R} \subset \mathbb{M}_{n_0}$ from \mathcal{J} of Lemma 6.10, and set $x_{\mu, b}^{(L)} = \hat{x}_{\mu, b}^{(L)}$ and $x_{\mu, a}^{(R)} = \hat{x}_{\mu, a}^{(R)}$.

If $k_R \in \mathbb{N}$, we define

$$D_a := \sum_{i, j=-k_R}^0 \left(\hat{D}_a^{(R)} \right)_{-i, -j} E_{i, j}^{(k_R, 0)}, \quad a = 1, \dots, k_R.$$

As $\hat{D}_a^{(R)} \in \text{DT}_{0, k_R+1}$, we have $D_a \in \text{UT}_{0, k_R+1}$. We have

$$D_a E_{00}^{(k_R, 0)} = \sum_{i=-k_R}^0 \left(\hat{D}_a^{(R)} \right)_{-i, 0} E_{i, 0}^{(k_R, 0)} = \sum_{i=-k_R}^0 \left(\hat{D}_a^{(R)} E_{00}^{(0, k_R)} \right)_{-i, 0} E_{i, 0}^{(k_R, 0)} = \sum_{i=-k_R}^0 \left(E_{a0}^{(0, k_R)} \right)_{-i, 0} E_{i, 0}^{(k_R, 0)} = E_{-a, 0}^{(k_R, 0)}.$$

By 7 of Lemma 6.10, The linear span of $\{D_a\}_{a=1}^{k_R}$ is a subalgebra of UT_{0, k_R+1} . Therefore, $\mathbb{D} = (D_1, \dots, D_{k_R})$ belongs to $\mathcal{C}^R(k_R)$. If $k_L \in \mathbb{N}$, we set $G_b := \hat{D}_b^{(L)}$, $b = 1, \dots, k_L$. That $\mathbb{G} \in \mathcal{C}^L(k_L)$ follows from the corresponding properties of $\hat{\mathbb{D}}^{(L)} = (\hat{D}_1^{(L)}, \dots, \hat{D}_{k_L}^{(L)})$.

We define $\lambda = (\lambda_{-k_R}, \dots, \lambda_{k_L}) \in \mathbb{C}^{k_L+k_R+1}$ by

$$\lambda_i := \begin{cases} \hat{\lambda}_{-i}^{(R)}, & i = -k_R, \dots, 0 \\ \hat{\lambda}_i^{(L)}, & i = 1, \dots, k_L \end{cases}.$$

Then, by (56), we have $\lambda \in \text{Wo}(k_R, k_L)$.

Let

$$\tilde{Y}_R := \sum_{i, j=-k_R}^0 \left(\hat{Y}_R \right)_{-i, -j} E_{i, j}^{(k_R, 0)}.$$

Then we define

$$Y := I_R^{(k_R, k_L)} (\tilde{Y}_R) + I_L^{(k_R, k_L)} (\hat{Y}_L) \in \text{UT}_{0, k_R + k_L + 1}.$$

The properties 1, 4, 8 of Lemma 6.10 guarantees that $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$.

Now we prove that our $n_0, k_L, k_R, \omega, (\lambda, \mathbb{D}, \mathbb{G}, Y), l_0$, and $\{x_{\mu, b}^{(L)}\}, \{x_{\mu, a}^{(R)}\}$ satisfy conditions 1-4 of the Lemma. First, note that

$$\begin{aligned} B_{\mu^{(l)}} \hat{P}_R^{(n_0, k_R, k_L)} &= \left(id \otimes I_R^{(k_R, k_L)} \right) \left(\sum_{i, j = -k_R}^0 \left(\hat{v}_{\mu^{(l)}}^{(R)} \right)_{-i, -j} E_{i, j}^{(k_R, 0)} \right), \\ \hat{P}_L^{(n_0, k_R, k_L)} B_{\mu^{(l)}} &= \left(id \otimes I_L^{(k_R, k_L)} \right) \left(\hat{v}_{\mu^{(l)}}^{(L)} \right), \end{aligned} \quad (62)$$

for all $l \in \mathbb{N}$, and $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$. This and 9 of Lemma 6.10 implies 1. Furthermore, (62) and 10 of Lemma 6.10 implies 2, 3. 4 is 11 of Lemma 6.10. \square

Next we would like to show that for this $\mathbb{B}, \mathcal{K}_l(\mathbb{B})$ coincides with $M_{n_0} \otimes \left(\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_\lambda (1 + Y))^l \right)$ for l large enough. Note that we have

$$\begin{aligned} \mathcal{V}_l &:= M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_\lambda (1 + Y))^l \\ &= M_{n_0} \otimes \text{span} \left((\Lambda_\lambda (1 + Y))^l, \{ I_R^{(k_R, k_L)} (D_a) (\Lambda_\lambda (1 + Y))^l \}_{a=1}^{k_R} \cup \{ (\Lambda_\lambda (1 + Y))^l I_L^{(k_R, k_L)} (G_b) \}_{b=1}^{k_L} \right) + \text{CN}(n_0, k_R, k_L). \end{aligned} \quad (63)$$

First we show the following inclusion.

Lemma 7.3. Assume [A1], [A3], [A4], and [A5]. Recall $n_0, k_L, k_R, \omega, (\lambda, \mathbb{D}, \mathbb{G}, Y), l_0$, and $\{x_{\mu, b}^{(L)}\}, \{x_{\mu, a}^{(R)}\}$ given in Lemma 7.2, and \mathbb{B} defined by (61). We use the notation $\check{\Lambda} := \Lambda_\lambda (1 + Y)$. Then we have

$$\mathcal{K}_l(\mathbb{B}) \subset M_{n_0} \otimes \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \check{\Lambda}^l, \quad l \in \mathbb{N}. \quad (64)$$

In particular, for any $l_0 \leq l, \alpha, \beta = 1, \dots, n_0, a = 0, \dots, k_R, b = 0, \dots, k_L$, there exist $A_{\alpha\beta a}^{(R, l)}, A_{\alpha\beta b}^{(L, l)} \in \mathcal{K}_l(\mathbb{B})$ of the form

$$\begin{aligned} A_{\alpha\beta 0}^{(R, l)} &= e_{\alpha\beta}^{(n_0)} \otimes \check{\Lambda}^l + \sum_{b=1}^{k_L} z_{\alpha\beta 0b}^{(R, l)} \otimes \check{\Lambda}^l I_L^{(k_R, k_L)} (G_b) + O_{\alpha, \beta, 0}^{(R, l)} \\ A_{\alpha\beta a}^{(R, l)} &= e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)} (D_a) \check{\Lambda}^l + \sum_{b=1}^{k_L} z_{\alpha\beta ab}^{(R, l)} \otimes \check{\Lambda}^l I_L^{(k_R, k_L)} (G_b) + O_{\alpha, \beta, a}^{(R, l)} \\ A_{\alpha\beta 0}^{(L, l)} &= e_{\alpha\beta}^{(n_0)} \otimes \check{\Lambda}^l + \sum_{a=1}^{k_R} z_{\alpha\beta a0}^{(L, l)} \otimes I_R^{(k_R, k_L)} (D_a) \check{\Lambda}^l + O_{\alpha, \beta, 0}^{(L, l)} \\ A_{\alpha\beta b}^{(L, l)} &= e_{\alpha\beta}^{(n_0)} \otimes \check{\Lambda}^l I_L^{(k_R, k_L)} (G_b) + \sum_{a=1}^{k_R} z_{\alpha\beta ab}^{(L, l)} \otimes I_R^{(k_R, k_L)} (D_a) \check{\Lambda}^l + O_{\alpha, \beta, b}^{(L, l)}. \end{aligned} \quad (65)$$

Here, $z_{\alpha\beta ab}^{(\sigma, l)} \in M_{n_0}$ and $O_{\alpha, \beta, k}^{(\sigma, l)} \in \text{CN}(n_0, k_R, k_L)$.

Proof. From (16) of Part I, it is easy to check $\mathcal{V}_{l_1} \mathcal{V}_{l_2} \subset \mathcal{V}_{l_1 + l_2}$. As $\mathcal{K}_1(\mathbb{B}) \subset \mathcal{V}_1$, this proves the first claim of the Lemma, inductively. The second claim can be checked from the first one and 1 of Lemma 7.2. \square

Next we prove the opposite inclusion for large l , inductively. The following Lemma is used for the induction.

Lemma 7.4. *Assume $[A1], [A3], [A4]$, and $[A5]$. We use the notations in Lemma 7.2 and $\check{\Lambda} := \Lambda_\lambda(1+Y)$. For each $a = 0, \dots, k_R$ and $l \in \mathbb{N}$, set*

$$W_a^{(l)} := M_{n_0} \otimes \text{span} \left\{ I_R^{(k_R, k_L)}(D_{a'}) \check{\Lambda}^l + \text{CN}(n_0, k_R, k_L) \right\}.$$

We consider the following two propositions, for $a = 0, \dots, k_R$.

(\mathbb{P}_a) : *There exists an $l_{a,R} \in \mathbb{N}$ satisfying the following: for any $l_{a,R} \leq l \in \mathbb{N}$, there exist $X_{\alpha\beta a'}^{(l)} \in \mathcal{K}_l(\mathbb{B})$, $\alpha, \beta = 1, \dots, n_0$, $a' = 0, \dots, a$ such that*

$$\begin{aligned} X_{\alpha\beta 0}^{(l)} - e_{\alpha\beta}^{(n_0)} \otimes \check{\Lambda}^l &\in W_a^{(l)}, \\ X_{\alpha\beta a'}^{(l)} - e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_{a'}) \check{\Lambda}^l &\in W_a^{(l)}, \quad 1 \leq a' \leq a. \end{aligned}$$

$(\tilde{\mathbb{P}}_a)$: *There exist $\tilde{l}_{a,R} \in \mathbb{N}$ and $Z \in \mathcal{K}_{\tilde{l}_{a,R}}(\mathbb{B})$ with the form*

$$Z = \sum_{a' \geq a+1} z_{a'} \otimes I_R^{(k_R, k_L)}(D_{a'}) \check{\Lambda}^{\tilde{l}_{a,R}} + \text{an element in } \text{CN}(n_0, k_R, k_L),$$

with

$$z_{a+1} \neq 0.$$

If (\mathbb{P}_a) and $(\tilde{\mathbb{P}}_a)$ hold for some $a < k_R$, then (\mathbb{P}_{a+1}) holds.

Proof. First we note the following properties which can be checked from Definition 1.7, 1.8, Remark 1.9 of Part I.

$$\begin{aligned} W_a^{(l_1)} (M_{n_0} \otimes \check{\Lambda}^{l_2}), (M_{n_0} \otimes \check{\Lambda}^{l_2}) W_a^{(l_1)} &\subset W_a^{(l_1+l_2)} \\ W_a^{(l_1)} (M_{n_0} \otimes I_R^{(k_R, k_L)}(D_b) \check{\Lambda}^{l_2}), (M_{n_0} \otimes I_R^{(k_R, k_L)}(D_b) \check{\Lambda}^{l_2}) W_a^{(l_1)} &\subset W_{a+1}^{(l_1+l_2)} \\ W_a^{(l_1)} \text{CN}(n_0, k_R, k_L) &\subset \text{CN}(n_0, k_R, k_L), \quad \text{CN}(n_0, k_R, k_L) W_a^{(l_1)} = 0, \quad \text{CN}(n_0, k_R, k_L) \cdot \text{CN}(n_0, k_R, k_L) = 0, \\ W_a^{(l_1)} \cdot W_{a'}^{(l_2)} &\subset W_{\max\{a, a'\}+1}^{(l_1+l_2)}, \quad \hat{P}_L^{(n_0, k_R, k_L)} W_b^{(l)} = 0, \end{aligned} \tag{66}$$

for all $l_1, l_2 \in \mathbb{N}$, $a, a' = 0, \dots, k_R$, and $b = 1, \dots, k_R$. Now, assume that (\mathbb{P}_a) and $(\tilde{\mathbb{P}}_a)$ hold for some $a < k_R$. Then using (66), we get

$$\begin{aligned} e_{\alpha_1\beta_1}^{(n_0)} z_{a+1} e_{\alpha_2\beta_2}^{(n_0)} \otimes \check{\Lambda}^{l_1} I_R^{(k_R, k_L)}(D_{a+1}) \check{\Lambda}^{\tilde{l}_{a,R}} \check{\Lambda}^{l_2} &= X_{\alpha_1\beta_1 0}^{(l_1)} Z X_{\alpha_2\beta_2 0}^{(l_2)} + \text{an element in } W_{a+1}^{(l_1+l_2+\tilde{l}_{a,R})} \\ &\in \mathcal{K}_{l_1+l_2+\tilde{l}_{a,R}}(\mathbb{B}) + W_{a+1}^{(l_1+l_2+\tilde{l}_{a,R})}, \end{aligned}$$

for any $l_1, l_2 \geq l_{a,R}$, $\alpha_1, \beta_1, \alpha_2, \beta_2 = 1, \dots, n_0$. Further calculation using (9)-(13) of Part I shows that

$$\begin{aligned} e_{\alpha_1\beta_1}^{(n_0)} z_{a+1} e_{\alpha_2\beta_2}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_{a+1}) \check{\Lambda}^{l_1+l_2+\tilde{l}_{a,R}} \\ = \lambda_{-a-1}^{-l_1} e_{\alpha_1\beta_1}^{(n_0)} z_{a+1} e_{\alpha_2\beta_2}^{(n_0)} \otimes \check{\Lambda}^{l_1} I_R^{(k_R, k_L)}(D_{a+1}) \check{\Lambda}^{\tilde{l}_{a,R}} \check{\Lambda}^{l_2} + \text{an element in } W_{a+1}^{(l_1+l_2+\tilde{l}_{a,R})} \\ \in \mathcal{K}_{l_1+l_2+\tilde{l}_{a,R}}(\mathbb{B}) + W_{a+1}^{(l_1+l_2+\tilde{l}_{a,R})}. \end{aligned}$$

As $z_{a+1} \neq 0$, there exists α_2, β_1 such that $\langle \chi_{\beta_1}^{(n_0)}, z_{a+1} \chi_{\alpha_2}^{(n_0)} \rangle \neq 0$. Hence we obtain

$$e_{\alpha_1 \beta_2}^{(n_0)} \otimes I_R^{(k_R, k_L)} (D_{a+1}) \check{\Lambda}^{l_1 + l_2 + \tilde{l}_{a,R}} \in \mathcal{K}_{l_1 + l_2 + \tilde{l}_{a,R}}(\mathbb{B}) + W_{a+1}^{(l_1 + l_2 + \tilde{l}_{a,R})}, \quad l_1, l_2 \geq l_{a,R}, \quad \alpha_1, \beta_2 = 1, \dots, n_0. \quad (67)$$

From this, we get

$$W_a^{(l)} \subset \mathcal{K}_l(\mathbb{B}) + W_{a+1}^{(l)}, \quad 2l_{a,R} + \tilde{l}_{a,R} \leq l.$$

Therefore, from (\mathbb{P}_a) , we obtain

$$e_{\alpha\beta}^{(n_0)} \otimes \check{\Lambda}^l, e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)} (D_{a'}) \check{\Lambda}^l \in \mathcal{K}_l(\mathbb{B}) + W_a^{(l)} \subset \mathcal{K}_l(\mathbb{B}) + W_{a+1}^{(l)}$$

for any $1 \leq a' \leq a$, $2l_{a,R} + \tilde{l}_{a,R} \leq l$, $\alpha, \beta = 1, \dots, n_0$. This and (67) implies (\mathbb{P}_{a+1}) with $l_{a+1,R} = 2l_{a,R} + \tilde{l}_{a,R}$. \square

In order to apply Lemma 7.4, we have to find Z as in $(\tilde{\mathbb{P}}_a)$. In the proof of the following Lemma, we use Lemma C.7 of Part I to find such Z .

Lemma 7.5. *Assume $[A1], [A3], [A4]$, and $[A5]$. We use the notations in Lemma 7.2. Assume that (\mathbb{P}_a) holds for some $a < k_R$. Then (\mathbb{P}_{a+1}) holds.*

Proof. For $X_{\alpha\beta a'}^{(l)}$ given in (\mathbb{P}_a) , we define $\tilde{X}_{\mathbb{I}}^{(l)} := \sum_{\alpha=1}^{n_0} X_{\alpha\alpha 0}^{(l)} \in \mathcal{K}_l(\mathbb{B})$, $l_{a,R} \leq l$. We have

$$\tilde{X}_{\mathbb{I}}^{(l)} - \mathbb{I} \otimes \check{\Lambda}^l \in W_a^{(l)}.$$

Set $l'_0 := l_{a,R} + k_L + k_R + 1$. For $A_{\alpha\beta a+1}^{(R, l_0)}$ in Lemma 7.3, using (66), we obtain

$$\begin{aligned} & \tilde{X}_{\mathbb{I}}^{(j+l'_0)} A_{\alpha\beta a+1}^{(R, l_0)} \tilde{X}_{\mathbb{I}}^{(l-j-l'_0-l_0)} \\ &= e_{\alpha\beta}^{(n_0)} \otimes \check{\Lambda}^{j+l'_0} I_R^{(k_R, k_L)} (D_{a+1}) \check{\Lambda}^{l-j-l'_0} + \sum_{b=1}^{k_L} z_{\alpha, \beta, a+1, b}^{(R, l_0)} \otimes \check{\Lambda}^{j+l'_0+l_0} I_L^{(k_R, k_L)} (G_b) \check{\Lambda}^{l-j-l'_0-l_0} \\ &+ \text{an element in } W_{a+1}^l \\ &= \lambda_{-a-1}^{j+l'_0} \cdot e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)} (D_{a+1}) \check{\Lambda}^l + \sum_{b=1}^{k_L} \sum_{b'=1}^{k_L} \left\langle f_b^{(k_R, k_L)}, \check{\Lambda}^{l-j-l'_0-l_0} f_{b'}^{(k_R, k_L)} \right\rangle z_{\alpha, \beta, a+1, b}^{(R, l_0)} \otimes \check{\Lambda}^l I_L^{(k_R, k_L)} (G_{b'}), \\ &+ \text{an element in } W_{a+1}^l, \end{aligned}$$

for all $l \geq 2(k_L + k_R + 1)^2 + l_0 + 2l'_0$, and $0 \leq j \leq (k_L + k_R + 1)^2$. Here, for the second equality, we used (12), (13) and Remark 1.9 of Part I. Note that we have $\left\langle f_b^{(k_R, k_L)}, \check{\Lambda}^{l-j-l'_0-l_0} f_{b'}^{(k_R, k_L)} \right\rangle = \lambda_b^{l-l'_0-l_0-j} \sum_{\gamma=0}^{k_L+k_R} l-l'_0-l_0-j C_\gamma \left\langle f_b^{(k_R, k_L)}, Y^\gamma f_{b'}^{(k_R, k_L)} \right\rangle$. Hence, from Lemma C.7 of Part I, there exists $\xi \in \mathbb{C}^{(k_L+k_R+1)^2}$ such that

$$\mathcal{K}_{\tilde{l}_{a,R}}(\mathbb{B}) \ni \sum_{j=0}^{(k_L+k_R+1)^2-1} \xi(j) \tilde{X}_{\mathbb{I}}^{(j+l'_0)} A_{\alpha\beta a+1}^{(R, l_0)} \tilde{X}_{\mathbb{I}}^{(\tilde{l}_{a,R}-j-l'_0-l_0)} = e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)} (D_{a+1}) \check{\Lambda}^{\tilde{l}_{a,R}} + \text{an element in } W_{a+1}^{\tilde{l}_{a,R}}.$$

where $\tilde{l}_{a,R} = 2(k_L + k_R + 1)^2 + l_0 + 2l'_0$. In other words, $(\tilde{\mathbb{P}}_a)$ holds. Applying Lemma 7.4, (\mathbb{P}_{a+1}) holds. \square

With this induction step, we obtain the following Lemma.

Lemma 7.6. *Assume [A1],[A3],[A4], and [A5]. We use the notations in Lemma 7.2 and $\check{\Lambda} := \Lambda_{\lambda}(1+Y)$. Then there exists an $l'_0 \in \mathbb{N}$ such that*

$$M_{n_0} \otimes \text{span} \left(\check{\Lambda}^l, \{I_R^{(k_R, k_L)}(D_a) \check{\Lambda}^l\}_{a=1}^{k_R} \cup \{\check{\Lambda}^l I_L^{(k_R, k_L)}(G_b)\}_{b=1}^{k_L} \right) \subset \mathcal{K}_l(\mathbb{B}) + \text{CN}(n_0, k_R, k_L), \quad (68)$$

for any $l'_0 \leq l \in \mathbb{N}$.

Proof. (\mathbb{P}_{k_R}) combined with Lemma 7.3 (65) corresponds to the claim. From Lemma 7.5, it suffices to check (\mathbb{P}_0) . However, this is clear from the existence of $A_{\alpha\beta 0}^{(L, l)}$ in Lemma 7.3. \square

Lemma 7.7. *Assume [A1],[A3],[A4], and [A5]. We use the notations in Lemma 7.2. Then there exists an $\tilde{l}_0 \geq l'_0$ (with l'_0 given in Lemma 7.6) such that*

$$\text{CN}(n_0, k_R, k_L) \subset \mathcal{K}_l(\mathbb{B}), \quad (69)$$

for any $\tilde{l}_0 \leq l \in \mathbb{N}$.

Proof. First we claim

$$\begin{aligned} I_R^{(k_R, k_L)}(D_a) \overline{P_L^{(k_R, k_L)}} &= Q_{R, -(a+1)}^{(k_R, k_L)} I_R^{(k_R, k_L)}(D_a) \overline{P_L^{(k_R, k_L)}}, \quad a = 1, \dots, k_R, \\ \overline{P_R^{(k_R, k_L)}} I_L^{(k_R, k_L)}(G_b) &= \overline{P_R^{(k_R, k_L)}} I_L^{(k_R, k_L)}(G_b) Q_{L, b+1}^{(k_R, k_L)}, \quad b = 1, \dots, k_L. \end{aligned} \quad (70)$$

To see this, let $-k_R \leq i \leq 0$ be a number satisfying $E_{ii}^{(k_R, k_L)} I_R^{(k_R, k_L)}(D_a) \overline{P_L^{(k_R, k_L)}} \neq 0$ for $a = 1, \dots, k_R$. As we have

$$I_R^{(k_R, k_L)} \left(E_{ii}^{(k_R, 0)} D_a \sum_{j=-k_R}^{-1} E_{jj}^{(k_R, 0)} \right) = E_{ii}^{(k_R, k_L)} I_R^{(k_R, k_L)}(D_a) \overline{P_L^{(k_R, k_L)}} \neq 0,$$

it means there is $j \in \{-k_R, \dots, -1\}$ such that $E_{ii}^{(k_R, 0)} D_a E_{jj}^{(k_R, 0)} \neq 0$. By Definition 1.8 (9) in Part I, this implies $\lambda_i = \lambda_{-a} \lambda_j$. In particular, we have $|\lambda_i| = |\lambda_{-a} \lambda_j| < |\lambda_{-a}|$, because $|\lambda_j| < 1$ for $j \in \{-k_R, \dots, -1\}$. As $\lambda \in \text{Wo}(k_R, k_L)$, this implies $i \leq -a - 1$. This proves the first line of the claim. The second one can be proven similarly.

Let $\check{\Lambda} := \Lambda_{\lambda}(1+Y)$ as before. Note that

$$\begin{aligned} I_R^{(k_R, k_L)}(D_a) \check{\Lambda}^{l_1+l_2} I_L^{(k_R, k_L)}(G_b) &= I_R^{(k_R, k_L)}(D_a) P_R^{(k_R, k_L)} P_L^{(k_R, k_L)} \check{\Lambda}^{l_1+l_2} I_L^{(k_R, k_L)}(G_b) \\ &= I_R^{(k_R, k_L)}(D_a) E_{00}^{(k_R, k_L)} \check{\Lambda}^{l_1+l_2} I_L^{(k_R, k_L)}(G_b) = I_R^{(k_R, k_L)}(D_a) E_{00}^{(k_R, k_L)} I_L^{(k_R, k_L)}(G_b) \\ &= I_R^{(k_R, k_L)} \left(D_a E_{00}^{(k_R, 0)} \right) I_L^{(k_R, k_L)} \left(E_{00}^{(0, k_L)} G_b \right) = I_R^{(k_R, k_L)} \left(E_{-a0}^{(k_R, 0)} \right) I_L^{(k_R, k_L)} \left(E_{0b}^{(0, k_L)} \right) = E_{-ab}^{(k_R, k_L)}, \end{aligned} \quad (71)$$

for any $l_1, l_2 \in \mathbb{N}$ and $a = 1, \dots, k_R$, $b = 1, \dots, k_L$. By the claim (70), we have

$$\begin{aligned} Z \check{\Lambda}^{l_2} I_L^{(k_R, k_L)}(G_b) &= Z \check{\Lambda}^{l_2} \overline{P_R^{(k_R, k_L)}} I_L^{(k_R, k_L)}(G_b) = Z \check{\Lambda}^{l_2} \overline{P_R^{(k_R, k_L)}} I_L^{(k_R, k_L)}(G_b) Q_{L, b+1}^{(k_R, k_L)} \in \overline{P_L^{(k_R, k_L)}} M_{k_L+k_R+1} Q_{L, b+1}^{(k_R, k_L)}, \\ Z \in \overline{P_L^{(k_R, k_L)}} M_{k_L+k_R+1} \overline{P_R^{(k_R, k_L)}}, \quad b = 1, \dots, k_L, \quad l_2 \in \mathbb{N}, \end{aligned} \quad (72)$$

and

$$I_R^{(k_R, k_L)}(D_a) \check{\Lambda}^{l_1} Z = I_R^{(k_R, k_L)}(D_a) \overline{P_L^{(k_R, k_L)}} \check{\Lambda}^{l_1} Z = Q_{R, -a-1}^{(k_R, k_L)} I_R^{(k_R, k_L)}(D_a) \overline{P_L^{(k_R, k_L)}} \check{\Lambda}^{l_1} Z \in Q_{R, -a-1}^{(k_R, k_L)} M_{k_L + k_R + 1} Q_{L, b}^{(k_R, k_L)},$$

$$Z \in \overline{P_L^{(k_R, k_L)}} M_{k_L + k_R + 1} Q_{L, b}^{(k_R, k_L)}, \quad a = 1, \dots, k_R, b = 1, \dots, k_L, \quad l_1 \in \mathbb{N}. \quad (73)$$

For $a = 1, \dots, k_R$ and $b = 1, \dots, k_L$, we define the subset $\mathcal{M}_{a,b}$ of $\{1, \dots, k_R\} \times \{1, \dots, k_L\}$ by

$$\mathcal{M}_{a,b} := \{(a', b') \mid a' = 1, \dots, k_R, \quad 1 \leq b' < b\} \cup \{(a', b') \mid 1 \leq a' \leq a, \quad b' = b\}.$$

We also set

$$\mathcal{P}_{a,b} := M_{n_0} \otimes Q_{R, -a-1}^{(k_R, k_L)} M_{k_L + k_R + 1} Q_{L, b}^{(k_R, k_L)} + M_{n_0} \otimes \overline{P_L^{(k_R, k_L)}} M_{k_L + k_R + 1} Q_{L, b+1}^{(k_R, k_L)}.$$

Note that

$$M_{n_0} \otimes E_{-a', b'}^{(k_R, k_L)} \subset \mathcal{P}_{a,b}, \quad (a', b') \in (\{1, \dots, k_R\} \times \{1, \dots, k_L\}) \setminus \mathcal{M}_{a,b}.$$

For $(a, b) \in \{1, \dots, k_R\} \times \{1, \dots, k_L\}$, we consider the following proposition:

$(\mathfrak{P}_{a,b})$: There exists an $l_{a,b} \in \mathbb{N}$ with $l'_0 \leq l_{a,b}$ such that

$$M_{n_0} \otimes E_{-a', b'}^{(k_R, k_L)} \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{a,b}, \quad (74)$$

for all $(a', b') \in \mathcal{M}_{a,b}$ and $l \geq l_{a,b}$.

First we show \mathfrak{P}_{11} . By Lemma 7.6, for any $l'_0 \leq l_1, l_2 \in \mathbb{N}$ and $\alpha, \beta = 1, \dots, n_0$, there exist $Z_{1, \alpha, \beta}^{(l_1)}, Z_{2, \beta, \beta}^{(l_2)} \in \text{CN}(n_0, k_R, k_L)$ such that

$$e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_1) \check{\Lambda}^{l_1} + Z_{1, \alpha, \beta}^{(l_1)} \in \mathcal{K}_{l_1}(\mathbb{B}), \quad e_{\beta\beta}^{(n_0)} \otimes \check{\Lambda}^{l_2} I_L^{(k_R, k_L)}(G_1) + Z_{2, \beta, \beta}^{(l_2)} \in \mathcal{K}_{l_2}(\mathbb{B}).$$

By (66), we have

$$\begin{aligned} \mathcal{K}_{l_1 + l_2}(\mathbb{B}) &\supset \left(e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_1) \check{\Lambda}^{l_1} + Z_{1, \alpha, \beta}^{(l_1)} \right) \left(e_{\beta\beta}^{(n_0)} \otimes \check{\Lambda}^{l_2} I_L^{(k_R, k_L)}(G_1) + Z_{2, \beta, \beta}^{(l_2)} \right) \\ &= e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_1) \check{\Lambda}^{l_1 + l_2} I_L^{(k_R, k_L)}(G_1) + Z_{1, \alpha, \beta}^{(l_1)} \left(e_{\beta\beta}^{(n_0)} \otimes \check{\Lambda}^{l_2} I_L^{(k_R, k_L)}(G_1) \right) + \left(e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_1) \check{\Lambda}^{l_1} \right) Z_{2, \beta, \beta}^{(l_2)}. \end{aligned} \quad (75)$$

From (71), (72), and (73), this implies $e_{\alpha\beta}^{(n_0)} \otimes E_{-11}^{(k_R, k_L)} \in \mathcal{P}_{11} + \mathcal{K}_l(\mathbb{B})$, for any $\alpha, \beta = 1, \dots, n_0$ and $l \geq 2l'_0$. This proves $(\mathfrak{P}_{1,1})$.

Assume that $(\mathfrak{P}_{a,b})$ holds for some $a < k_R$ and $b = 1, \dots, k_L$. We would like to show that $(\mathfrak{P}_{a+1,b})$ holds. By $(\mathfrak{P}_{a,b})$, we have

$$\text{CN}(n_0, k_R, k_L) \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{a,b}, \quad l \geq l_{a,b}.$$

This and Lemma 7.6 implies

$$M_{n_0} \otimes I_R^{(k_R, k_L)}(D_{a+1}) \check{\Lambda}^l, M_{n_0} \otimes \check{\Lambda}^l I_L^{(k_R, k_L)}(G_b) \subset \mathcal{K}_l(\mathbb{B}) + \text{CN}(n_0, k_R, k_L) \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{a,b}, \quad l \geq l_{a,b}.$$

Therefore, for any $l_{a,b} \leq l_1, l_2 \in \mathbb{N}$ and $\alpha, \beta = 1, \dots, n_0$, there exist $Z_{1, \alpha, \beta}^{(l_1)}, Z_{2, \beta, \beta}^{(l_2)} \in \mathcal{P}_{a,b}$ such that

$$e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_{a+1}) \check{\Lambda}^{l_1} + Z_{1, \alpha, \beta}^{(l_1)} \in \mathcal{K}_{l_1}(\mathbb{B}), \quad e_{\beta\beta}^{(n_0)} \otimes \check{\Lambda}^{l_2} I_L^{(k_R, k_L)}(G_b) + Z_{2, \beta, \beta}^{(l_2)} \in \mathcal{K}_{l_2}(\mathbb{B}).$$

From (71), (72), and (73), this implies $e_{\alpha\beta}^{(n_0)} \otimes E_{-(a+1), b}^{(k_R, k_L)} \in \mathcal{P}_{a+1, b} + \mathcal{K}_l(\mathbb{B})$, for any $\alpha, \beta = 1, \dots, n_0$ and $l \geq 2l_{a,b}$. Hence we have

$$M_{n_0} \otimes E_{-a', b'}^{(k_R, k_L)} \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{a,b} \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{a+1, b}$$

for all $(a', b') \in \mathcal{M}_{a+1, b}$ and $l \geq 2l_{a, b}$. This proves $(\mathfrak{P}_{a+1, b})$.

Assume that $(\mathfrak{P}_{k_R, b})$ holds for some $b < k_L$. We would like to show that $(\mathfrak{P}_{1, b+1})$ holds. By $(\mathfrak{P}_{k_R, b})$, we have

$$\text{CN}(n_0, k_R, k_L) \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{k_R, b}, \quad l \geq l_{k_R, b}.$$

This and Lemma 7.6 implies

$$\text{M}_{n_0} \otimes I_R^{(k_R, k_L)}(D_1) \check{\Lambda}^l, \text{M}_{n_0} \otimes \check{\Lambda}^l I_L^{(k_R, k_L)}(G_{b+1}) \subset \mathcal{K}_l(\mathbb{B}) + \text{CN}(n_0, k_R, k_L) \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{k_R, b}, \quad l \geq l_{k_R, b}.$$

Therefore, for any $l_{k_R, b} \leq l_1, l_2 \in \mathbb{N}$ and $\alpha, \beta = 1, \dots, n_0$, there exist $Z_{1, \alpha, \beta}^{(l_1)}, Z_{2, \beta, \beta}^{(l_2)} \in \mathcal{P}_{k_R, b}$ such that

$$e_{\alpha\beta}^{(n_0)} \otimes I_R^{(k_R, k_L)}(D_1) \check{\Lambda}^{l_1} + Z_{1, \alpha, \beta}^{(l_1)} \in \mathcal{K}_{l_1}(\mathbb{B}), \quad e_{\beta\beta}^{(n_0)} \otimes \check{\Lambda}^{l_2} I_L^{(k_R, k_L)}(G_{b+1}) + Z_{2, \beta, \beta}^{(l_2)} \in \mathcal{K}_{l_2}(\mathbb{B}).$$

From (71), (72), and (73), this implies $e_{\alpha\beta}^{(n_0)} \otimes E_{-1, b+1}^{(k_R, k_L)} \in \mathcal{P}_{1, b+1} + \mathcal{K}_l(\mathbb{B})$, for any $\alpha, \beta = 1, \dots, n_0$ and $l \geq 2l_{k_R, b}$. Hence we have

$$\text{M}_{n_0} \otimes E_{-a', b'}^{(k_R, k_L)} \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{k_R, b} \subset \mathcal{K}_l(\mathbb{B}) + \mathcal{P}_{1, b+1}$$

for all $(a', b') \in \mathcal{M}_{1, b+1}$ and $l \geq 2l_{k_R, b}$. This proves $(\mathfrak{P}_{1, b+1})$.

Hence we have proven $(\mathfrak{P}_{k_R, k_L})$, inductively. As $\mathcal{P}_{k_R, k_L} = \{0\}$ and $\mathcal{M}_{k_R, k_L} = \{1, \dots, k_R\} \times \{1, \dots, k_L\}$, we have

$$\text{M}_{n_0} \otimes E_{-a', b'}^{(k_R, k_L)} \subset \mathcal{K}_l(\mathbb{B}),$$

for all $(a', b') \in \{1, \dots, k_R\} \times \{1, \dots, k_L\}$ and $l \geq l_{k_R, k_L}$. This proves the Lemma. \square

Proof of Lemma 7.1. Let us consider the a septuplet $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ and ω given in Lemma 7.2. We define \mathbb{B} by 1 of Lemma 7.2. By Lemma 7.3, Lemma 7.6, and Lemma 7.7, our \mathbb{B} belongs to ClassA. The properties 2,3 of Lemma 7.1 corresponds to 2,3 of Lemma 7.2. By these properties, we have

$$\omega_L \circ \tau_x(h_{m', \mathbb{B}}) = 0, \quad x \leq -m', \quad \omega_R \circ \tau_x(h_{m', \mathbb{B}}) = 0, \quad 0 \leq x,$$

for $m' \geq m_{\mathbb{B}}$, because $\text{Ran } \Gamma_{l, \mathbb{B}}^{(R)}$ is a subspace of $\ker \tau_x^{(R)}(h_{m', \mathbb{B}})$ for $0 \leq x \leq l - m'$ and $l \geq m'$. Recall that if $\psi_\sigma \in \mathcal{S}_\sigma(H)$ for $\sigma = L, R$, by Lemma 2.3, we have $\psi_\sigma \leq d_1 \cdot \omega_\sigma$. Therefore, we have

$$\psi_L \circ \tau_x(h_{m', \mathbb{B}}) = 0, \quad x \leq -m', \quad \psi_R \circ \tau_x(h_{m', \mathbb{B}}) = 0, \quad 0 \leq x.$$

This means $\psi_L \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m', \mathbb{B}}})$ and $\psi_R \in \mathcal{S}_{[0, \infty)}(H_{\Phi_{m', \mathbb{B}}})$ for $m' \geq m_{\mathbb{B}}$. (Recall Lemma 2.1 and the fact that $h_{m', \mathbb{B}}$ satisfies [A1]-[A5].) This proves $\mathcal{S}_L(H) \subset \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m', \mathbb{B}}})$ and $\mathcal{S}_R(H) \subset \mathcal{S}_{[0, \infty)}(H_{\Phi_{m', \mathbb{B}}})$ for $m' \geq m_{\mathbb{B}}$. Conversely, let $\psi_L \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m', \mathbb{B}}})$ for $m' \geq m_{\mathbb{B}}$. Then, from Lemma 3.15 of Part I, there exists $\sigma_L \in \mathfrak{E}_{n_0(k_L+1)}$ such that $\psi_L(A) = \Xi_L(\sigma_L)(A) := \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} \mathbb{L}_{\mathbb{B}}(A) y_{\mathbb{B}}^{\frac{1}{2}})$. By the definition of $\mathbb{L}_{\mathbb{B}}$, it is easy to see that $\tau_l(s(\psi_L|_{\mathcal{A}_{[-l, -1]}}))$ is under the orthogonal projection onto $\Gamma_{l, \mathbb{B}}^{(R)} \left(\text{M}_{n_0} \otimes P_L^{(k_R, k_L)} \text{M}_{k_L + k_R + 1} P_L^{(k_R, k_L)} \right)$ for any $l \in \mathbb{N}$. By 2 of Lemma 7.2, this means that the support of $\psi_L|_{\mathcal{A}_{[-l, -1]}}$ is under $s(\omega_L|_{\mathcal{A}_{[-l, -1]}})$. As $s(\omega_L|_{\mathcal{A}_{[-l, -1]}})$ is under the projection onto the kernel of $\tau_x(h)$, for $-l \leq x \leq -m$ by Lemma 2.1, we obtain $\psi_L \circ \tau_x(h) = 0$. Hence we have $\psi_L \in \mathcal{S}_L(H)$. Hence we get $\mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m', \mathbb{B}}}) \subset \mathcal{S}_L(H)$. The proof for $\mathcal{S}_{[0, \infty)}(H_{\Phi_{m', \mathbb{B}}}) \subset \mathcal{S}_R(H)$ is the same.

To prove 4, note that ω_∞ is translation invariant because of the uniqueness of $\mathcal{S}_{\mathbb{Z}}(H)$. Note that $\omega_\infty|_{\mathcal{A}_{[0, \infty)}} \in \mathcal{S}_R(H) = \mathcal{S}_{[0, \infty)}(H_{\Phi_{m', \mathbb{B}}})$ because of 1 and Lemma 2.1. By the translation invariance of ω_∞ and Lemma 2.1, this means $\omega_\infty \in \mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m', \mathbb{B}}}) = \{\omega_{\mathbb{B}, \infty}\}$, $m' \geq m_{\mathbb{B}}$. Hence we have $\omega_\infty = \omega_{\mathbb{B}, \infty}$. \square

8 Proof of the main Theorem

In this section we complete the proof of Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2 . Let us consider the $\mathbb{B} \in \text{ClassA}$ with respect to the septuplet $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ given in Lemma 7.1. Recall the definition of $l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ from Part I Definition 1.13. We denote it by $l_{\mathbb{B}}$ throughout the proof.

We fix an arbitrary $m_1 \geq \max\{2l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y), \frac{\log(n_0^2(k_L+1)(k_R+1)+1)}{\log n}\}$. First we claim that there exist a constant $\tilde{C} > 0$ and a natural number $\tilde{N}_0 \in \mathbb{N}$ such that

$$|\text{Tr}_{[0, N-1]}((1 - G_{N, \mathbb{B}}) G_N)| \leq \tilde{C} s_1^{\frac{N}{2}}, \quad \tilde{N}_0 \leq N. \quad (76)$$

(Here, s_1 is given in [A4].) Recall Theorem 1.18 of Part I. By (ii) of the latter theorem, there exist $\gamma_{m_1, \mathbb{B}} > 0$ and $\tilde{N}_{m_1, \mathbb{B}} \in \mathbb{N}$ such that

$$\gamma_{m_1, \mathbb{B}} (1 - G_{N, \mathbb{B}}) \leq (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]}, \quad \text{for all } N \geq \tilde{N}_{m_1, \mathbb{B}}. \quad (77)$$

On the other hand, note that $\omega_R(\tau_x(h_{m_1, \mathbb{B}})) = 0$, $0 \leq x$ and $\omega_L(\tau_x(h_{m_1, \mathbb{B}})) = 0$, $x \leq -m_1$. This is because of Lemma 7.1, 1. Therefore, combining [A1] and [A4] with this, we obtain

$$|\text{Tr}_{[0, N-1]}(G_N(\tau_x(h_{m_1, \mathbb{B}})))| \leq d_1 C_1 s_1^{\max\{N-(m_1+x), x\}} \quad (78)$$

for all $0 \leq x \leq N - m_1$, and $N \geq N_3$. Set $\tilde{N}_0 := \max\{\tilde{N}_{m_1, \mathbb{B}}, N_3\}$. By (77) and (78), we obtain

$$\begin{aligned} \gamma_{m_1, \mathbb{B}} \text{Tr}_{[0, N-1]}((1 - G_{N, \mathbb{B}}) G_N) &\leq \text{Tr}_{[0, N-1]}((H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} G_N) \\ &= \sum_{0 \leq x \leq N-m_1} \text{Tr}_{[0, N-1]}(\tau_x(h_{m_1, \mathbb{B}}) G_N) \leq \sum_{0 \leq x \leq N-m_1} d_1 C_1 s_1^{\max\{N-(m_1+x), x\}} \end{aligned}$$

for all $N \geq \tilde{N}_0$. Therefore, there exists $\tilde{C} > 0$ such that

$$\text{Tr}_{[0, N-1]}((1 - G_{N, \mathbb{B}}) G_N) \leq \tilde{C} s_1^{\frac{N}{2}}, \quad N \geq \tilde{N}_0.$$

This proves the claim.

The same kind of inequality with G_N and $G_{N, \mathbb{B}}$ interchanged holds, i.e., there exist a constant $C' > 0$ and a natural number $N'_0 \in \mathbb{N}$ such that

$$|\text{Tr}_{[0, N-1]}((1 - G_N) G_{N, \mathbb{B}})| \leq C' s_{\mathbb{B}}^{\frac{N}{2}}, \quad N'_0 \leq N. \quad (79)$$

Here, $s_{\mathbb{B}}$ is given in (iv) of Theorem 1.18 Part I.

Define $0 < s < 1$, $C > 0$, and $N_0 \in \mathbb{N}$ by $s := \max\{s_1^{\frac{1}{4}}, s_{\mathbb{B}}^{\frac{1}{4}}\}$, $C := \tilde{C}^{\frac{1}{2}} + C'^{\frac{1}{2}}$, and $N_0 := \tilde{N}_0 + N'_0$. Then we have

$$\begin{aligned} \|G_N - G_{N, \mathbb{B}}\| &\leq \|G_N(\mathbb{I} - G_{N, \mathbb{B}})\| + \|(\mathbb{I} - G_N)G_{N, \mathbb{B}}\| \\ &\leq (\text{Tr}_{[0, N-1]}(G_N(\mathbb{I} - G_{N, \mathbb{B}})))^{\frac{1}{2}} + (\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}}(\mathbb{I} - G_N)))^{\frac{1}{2}} \leq \tilde{C}^{\frac{1}{2}} s_1^{\frac{N}{4}} + C'^{\frac{1}{2}} s_{\mathbb{B}}^{\frac{N}{4}} \leq C s^N, \end{aligned}$$

for all $N \geq N_0$. \square

Proof of Corollary 1.4. Let $\mathbb{B} \in \text{ClassA}$ and $m_1 \in \mathbb{N}$ given in Theorem 1.2. From [A2], Theorem 1.18 (ii) of Part I, and Theorem 1.2, there exists $\hat{\gamma} > 0$, $\tilde{N}_0 \in \mathbb{N}$, $C > 0$, and $0 < s < 1$ such that

$$\hat{\gamma}(1 - G_N) \leq (H)_{[0, N-1]}, \quad \hat{\gamma}(1 - G_{N, \mathbb{B}}) \leq (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]}, \quad \|G_N - G_{N, \mathbb{B}}\| \leq C s^N, \quad (80)$$

for all $\tilde{N}_0 \leq N \in \mathbb{N}$.

We claim the following: Let $\tilde{N}_0 \leq N \in \mathbb{N}$ and $t \in [0, 1]$. Assume that $\lambda \in (0, \hat{\gamma})$ is an eigenvalue of $(1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]}$ with a unit eigenvector $\xi \in \bigotimes_{i=0}^{N-1}$. Then we have

$$\|G_N \xi\| \leq \frac{C}{\lambda} \cdot N s^N, \quad \|G_{N, \mathbb{B}} \xi\| \leq \frac{C}{\lambda} \|h\| \cdot N s^N. \quad (81)$$

Multiplying

$$\left((1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \right) \xi = \lambda \xi, \quad (82)$$

by G_N , we obtain

$$t G_N (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \xi = \lambda G_N \xi.$$

From this, we obtain

$$\begin{aligned} \|G_N \xi\| &= \frac{t}{\lambda} \left\| G_N (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \xi \right\| \leq \frac{t}{\lambda} \left\| (G_N - G_{N, \mathbb{B}}) (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \xi \right\| + \frac{t}{\lambda} \left\| G_{N, \mathbb{B}} (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \xi \right\| \\ &= \frac{t}{\lambda} \left\| (G_N - G_{N, \mathbb{B}}) (H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \xi \right\| \leq \frac{t}{\lambda} \sum_{x: 0 \leq x \leq N-m_1} \|G_N - G_{N, \mathbb{B}}\| \leq \frac{tC}{\lambda} \cdot N s^N \leq \frac{C}{\lambda} \cdot N s^N. \end{aligned}$$

This proves the first inequality of the claim. The proof of the second one is the same.

Let $\tilde{N}_0 \leq N \in \mathbb{N}$ and $t \in [0, 1]$. Assume that $\lambda \in (0, \hat{\gamma})$ is an eigenvalue of $(1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]}$ with a unit eigenvector $\xi \in \bigotimes_{i=0}^{N-1}$. We have the following estimation on λ .

$$\hat{\gamma} \left(1 - \frac{C}{\lambda} (1 + \|h\|) N s^N \right) \leq \lambda. \quad (83)$$

To see this, we use the bound (80) (81) and obtain

$$\begin{aligned} \lambda &= \left\langle \xi, \left((1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \right) \xi \right\rangle \geq (1-t) \hat{\gamma} \langle \xi, (1-G_N) \xi \rangle + t \hat{\gamma} \langle \xi, (1-G_{N, \mathbb{B}}) \xi \rangle \\ &= \hat{\gamma} (1 - (1-t) \langle \xi, G_N \xi \rangle - t \langle \xi, G_{N, \mathbb{B}} \xi \rangle) \geq \hat{\gamma} \left(1 - (1-t) \frac{C}{\lambda} \cdot N s^N - t \frac{C}{\lambda} \|h\| \cdot N s^N \right) \geq \hat{\gamma} \left(1 - \frac{C}{\lambda} (1 + \|h\|) \cdot N s^N \right). \end{aligned}$$

Set $c_1 := \frac{2C}{\hat{\gamma}} (1 + \|h\|) \left(\sup_M M s^{\frac{M}{4}} \right)$. It is a positive constant. We also set $s_1 := s^{\frac{1}{2}}$ and $s_2 := s^{\frac{1}{4}}$. They satisfy $0 < s_1, s_2 < 1$. We claim

$$\sigma \left((1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]} \right) \cap [\hat{\gamma} c_1 s_1^N, \hat{\gamma} - \hat{\gamma} s_2^N] = \emptyset, \quad t \in [0, 1], \quad N \geq \hat{N}_0.$$

We prove this by contradiction. Assume this is not true. Then there exist $t \in [0, 1]$, $N \geq \hat{N}_0$, and $\lambda \in [\hat{\gamma} c_1 s_1^N, \hat{\gamma} - \hat{\gamma} s_2^N]$ such that λ is an eigenvalue of $(1-t)H_{[0, N-1]} + t(H_{\Phi_{m_1, \mathbb{B}}})_{[0, N-1]}$ with a unit eigenvector ξ . Then, by (83), we have

$$\hat{\gamma} \left(1 - \frac{C}{\lambda} (1 + \|h\|) N s^N \right) \leq \lambda \leq \hat{\gamma} (1 - s_2^N) = \hat{\gamma} (1 - s^{\frac{N}{4}}).$$

Comparing the left and the right hand side of this inequality, we obtain

$$\hat{\gamma} c_1 \leq \lambda s^{-\frac{N}{2}} \leq C N s^{\frac{N}{4}} (1 + \|h\|) \leq \frac{\hat{\gamma}}{2} c_1,$$

which is a contradiction. \square

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A Notations

In addition to the notations given in Subsection 1.1, 1.2, 1.3, and Appendix A of Part I, we use the following notations. For $\sigma = L, R$,

$$\begin{aligned}\tau_x^{(\sigma)} &:= \begin{cases} \tau_x, & \text{if } \sigma = R \\ \tau_{-x}, & \text{if } \sigma = L \end{cases}, \quad x \in \mathbb{N} \cup \{0\}, & \mathcal{S}_\sigma(H) &:= \begin{cases} \mathcal{S}_{[0, \infty)}(H), & \text{if } \sigma = R \\ \mathcal{S}_{(-\infty, -1]}(H), & \text{if } \sigma = L \end{cases}, \\ \mathcal{A}_\sigma &:= \begin{cases} \mathcal{A}_{[0, \infty)}, & \text{if } \sigma = R \\ \mathcal{A}_{(-\infty, -1]}, & \text{if } \sigma = L \end{cases}, & \mathcal{A}_{\sigma, l} &:= \begin{cases} \mathcal{A}_{[0, l-1]}, & \text{if } \sigma = R \\ \mathcal{A}_{[-l, -1]}, & \text{if } \sigma = L \end{cases}, \quad l \in \mathbb{N}, \\ \mathbb{Z}^{(\sigma)} &:= \begin{cases} \{x \in \mathbb{Z} \mid 0 \leq x\}, & \text{if } \sigma = R \\ \{x \in \mathbb{Z} \mid x \leq -m\}, & \text{if } \sigma = L \end{cases}, & \Gamma_\sigma &:= \begin{cases} [0, \infty), & \text{if } \sigma = R \\ (-\infty, -1], & \text{if } \sigma = L \end{cases}.\end{aligned}$$

Here $m \in \mathbb{N}$ in the definition of $\mathbb{Z}^{(\sigma)}$ is the interaction length of our h . Furthermore, for $\mu^{(l)} = (\mu_1, \dots, \mu_l) \in \{1, \dots, n\}^{\times l}$, $l \in \mathbb{N}$, we set

$$\mu^{(l, \sigma)} := \begin{cases} \mu^{(l)}, & \text{if } \sigma = R \\ (\mu_l, \mu_{l-1}, \dots, \mu_1), & \text{if } \sigma = L \end{cases}.$$

For n -tuple of $d \times d$ -matrices $\mathbf{v} = (v_1, \dots, v_n) \in M_d^{\times n}$ and $l \in \mathbb{N}$, we define $\Gamma_{l, \mathbf{v}}^{(\sigma)} : M_d \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ by

$$\Gamma_{l, \mathbf{v}}^{(\sigma)}(X) := \sum_{\mu^{(l)}} \left(\text{Tr } X \left(\widehat{v_{\mu^{(l, \sigma)}}} \right)^* \right) \widehat{\psi_{\mu^{(l)}}}, \quad X \in M_d.$$

B Endomorphisms of $B(\mathcal{H})$

Endomorphisms of $B(\mathcal{H})$ can be represented by a representation of Cuntz algebra.

Lemma B.1 ([A]). *Let \mathcal{H} be a separable infinite dimensional Hilbert space, and $n \in \mathbb{N}$. Let $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ be a unital endomorphism of $B(\mathcal{H})$ such that $(\Phi(B(\mathcal{H})))'$ is isomorphic to M_n . Then there exist $S_i \in B(\mathcal{H})$, $i = 1, \dots, n$ such that*

$$S_i^* S_j = \delta_{ij}, \quad \sum_{j=1}^n S_j x S_j^* = \Phi(x), \quad x \in B(\mathcal{H}). \quad (84)$$

From this Lemma, we obtain the following.

Lemma B.2. *Let $\mathfrak{A}, \mathfrak{B}$ be separable infinite dimensional simple unital C^* -algebras, and $n \in \mathbb{N}$, such that $\mathfrak{A} = M_n \otimes \mathfrak{B}$. Let ω be a pure state on \mathfrak{A} with GNS triple $(\mathcal{H}, \pi, \Omega)$. Let γ be a unital endomorphism of \mathfrak{A} such that $\gamma(\mathfrak{A}) = \mathbb{I} \otimes \mathfrak{B}$. Assume that ω and $\omega \circ \gamma$ are quasi-equivalent. Then there exist $S_i \in B(\mathcal{H})$, $i = 1, \dots, n$ such that*

$$S_i^* S_j = \delta_{ij}, \quad S_i S_j^* = \pi \left(e_{ij}^{(n)} \otimes \mathbb{I} \right), \quad \sum_{j=1}^n S_j \pi(A) S_j^* = \pi \circ \gamma(A), \quad A \in \mathfrak{A}.$$

Proof. As \mathfrak{A} is a separable infinite dimensional simple unital C^* -algebra, \mathcal{H} is a separable infinite dimensional Hilbert space. Note that $E : \pi(\mathfrak{A})'' \rightarrow \pi(M_n \otimes \mathbb{I})' \cap \pi(\mathfrak{A})''$,

$$E(x) := \sum_{i=1}^n \pi(e_{i1}^{(n)} \otimes \mathbb{I}) x \pi(e_{1i}^{(n)} \otimes \mathbb{I}),$$

defines a σw -continuous projection. By Lemma 2.6.8 of [BR1], we have $\pi(M_n \otimes \mathbb{I})' = \pi(\mathbb{I} \otimes \mathfrak{B})''$. This and the condition $\gamma(\mathfrak{A}) = \mathbb{I} \otimes \mathfrak{B}$ implies that $(\pi \circ \gamma(\mathfrak{A}))''$ is a factor. As ω and $\omega \circ \gamma$ are quasi-equivalent, ω is pure, and $(\pi \circ \gamma(\mathfrak{A}))''$ is a factor, there exists a σw -continuous unital endomorphism $\Phi : B(\mathcal{H}) \rightarrow B(\mathcal{H})$ such that $\Phi(\pi(A)) = \pi \circ \gamma(A)$, for $A \in \mathfrak{A}$, and $\Phi(B(\mathcal{H})) = (\pi \circ \gamma(\mathfrak{A}))''$. We have $(\Phi(B(\mathcal{H})))' = (\pi \circ \gamma(\mathfrak{A}))' = \pi(\mathbb{I} \otimes \mathfrak{B})' = \pi(M_n \otimes \mathbb{I})$. Applying Lemma B.1 we obtain S_i satisfying (84). As in the argument of Lemma 3.5 in [M1], we can deform S_i s so that they satisfy $S_i S_j^* = \pi(e_{ij}^{(n)} \otimes \mathbb{I})$. \square

C Finitely correlated states

First we recall the definitions introduced in [FNW2].

Definition C.1. Let $n \in \mathbb{N}$. The triple $(\mathfrak{B}, \mathbb{E}, \rho)$ given by a finite dimensional C^* -algebra \mathfrak{B} , a unital CP map $\mathbb{E} : M_n \otimes \mathfrak{B} \rightarrow \mathfrak{B}$, and a faithful state ρ on \mathfrak{B} such that $\rho \circ \mathbb{E}(\mathbb{I} \otimes X) = \rho(X)$, $X \in \mathfrak{B}$ is called a standard triple. For each $A \in M_n$, we define a map $\mathbb{E}_A : \mathfrak{B} \rightarrow \mathfrak{B}$ by $\mathbb{E}_A(X) = \mathbb{E}(A \otimes X)$, $X \in \mathfrak{B}$. A standard triple $(\mathfrak{B}, \mathbb{E}, \rho)$ is minimal if \mathfrak{B} has no proper sub C^* -algebra, which contains \mathbb{I} and is \mathbb{E}_A -invariant for any $A \in M_n$.

Definition C.2. Let $(\mathfrak{B}, \mathbb{E}, \rho)$ be a standard triple, and ω a state on $\mathcal{A}_{\mathbb{Z}}$. We say the standard triple $(\mathfrak{B}, \mathbb{E}, \rho)$ right (resp. left) generates ω if

$$\omega \left(\bigotimes_{i=a}^{a+l-1} A_i \right) = \rho \circ \mathbb{E}_{A_a} \circ \mathbb{E}_{A_{a+1}} \circ \cdots \circ \mathbb{E}_{A_{a+l-1}}(\mathbb{I}),$$

(resp.

$$\omega \left(\bigotimes_{i=a}^{a+l-1} A_i \right) = \rho \circ \mathbb{E}_{A_{a+l-1}} \circ \mathbb{E}_{A_{a+l-2}} \circ \cdots \circ \mathbb{E}_{A_a}(\mathbb{I}),)$$

for any $a \in \mathbb{Z}$, $l \in \mathbb{N}$, $A_i \in M_n$. If \mathfrak{B} is a $*$ -subalgebra of M_k containing unit \mathbb{I} and \mathbb{E} is given by an n -tuple of matrices $\mathbf{v} = (v_\mu)_{\mu=1}^n \subset M_{k+1}^{\times n}$ as

$$\mathbb{E}(e_{\mu\nu}^{(n)} \otimes X) := (v_\mu) X (v_\nu)^*, \quad X \in \mathfrak{B},$$

we also say that $(\mathfrak{B}, \mathbf{v}, \rho)$ right (resp. left) -generates ω . In this case, with a bit of abuse of notation, we say $(\mathfrak{B}, \mathbf{v}, \rho)$ is minimal if the corresponding $(\mathfrak{B}, \mathbb{E}, \rho)$ is minimal.

The formalism introduced in [FNW][FNW2] is the right version, but left version can be defined analogously.

For the class of states we consider, the minimal standard triple is unique up to isomorphism.

Theorem C.3. [FNW2] Let $n \in \mathbb{N}$ and ω be a state on $\mathcal{A}_{\mathbb{Z}}$. For each $N \in \mathbb{N}$, let D_N be the density matrix of $\omega|_{\mathcal{A}_{[0, N-1]}}$. Assume that $\sup_N \text{rank } D_N < \infty$. Let $(\mathfrak{B}_i, \mathbb{E}^{(i)}, \rho_i)$ $i = 1, 2$ be standard minimal triples right (resp. left) generating ω . Assume that the eigenspace of 1 for $\mathbb{E}_i^{(i)}$ is $\mathbb{C}\mathbb{I}_{\mathfrak{B}_i}$ for each $i = 1, 2$. Then there exists a C^* -isomorphism $\Theta : \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ such that $\mathbb{E}^{(2)} \circ (id_{M_n} \otimes \Theta) = \Theta \circ \mathbb{E}^{(1)}$.

From this, we can prove the following.

Lemma C.4. Let $n, k_1, k_2 \in \mathbb{N}$ and $\omega^{(i)} \in \text{Prim}_u(n, k_i)$. Let ρ_i be the faithful $T_{\omega^{(i)}}$ -invariant state. (See Lemma C.5 of Part I.) Let φ be a state on $\mathcal{A}_{\mathbb{Z}}$. For each $N \in \mathbb{N}$, let D_N be the density matrix of $\varphi|_{\mathcal{A}_{[0, N-1]}}$, and assume $\sup_N \text{rank } D_N < \infty$. Suppose that both of $(M_{k_1}, \omega^{(1)}, \rho_1)$ and $(M_{k_2}, \omega^{(2)}, \rho_2)$ right-generate φ . Then there exist a unitary $U : \mathbb{C}^{k_1} \rightarrow \mathbb{C}^{k_2}$ and $c \in \mathbb{T}$ such that

$$U\omega_{\mu}^{(1)} = c\omega_{\mu}^{(2)}U, \quad \mu = 1, \dots, n.$$

Proof. As $\omega^{(i)} \in \text{Prim}_u(n, k_i)$, $(M_{k_i}, \omega^{(i)}, \rho_i)$ is a minimal standard triple, for each $i = 1, 2$. Furthermore, the eigenspace of 1 for $T_{\omega^{(i)}}$ is $\mathbb{C}\mathbb{I}_{M_{k_i}}$, because of the primitivity of $\omega^{(i)}$. Therefore, we may apply Theorem C.3. By Theorem C.3, there is a $*$ -isomorphism $\Theta : M_{k_1} \rightarrow M_{k_2}$ such that

$$\omega_{\mu}^{(2)}\Theta(X)\left(\omega_{\nu}^{(2)}\right)^* = \Theta\left(\omega_{\mu}^{(1)}X\left(\omega_{\nu}^{(1)}\right)^*\right), \quad \mu, \nu = 1, \dots, n, \quad X \in M_{k_1}. \quad (85)$$

The rest of the proof is an easy case of Lemma 4.4. \square

D CP maps

Lemma D.1. Let $n \in \mathbb{N}$ and $T : M_n \rightarrow M_n$ be the irreducible unital CP map. Then the followings hold.

1. There exists $b \in \mathbb{N}$ such that $\sigma(T) \cap \mathbb{T} = \{\exp\left(\frac{2\pi i}{b}k\right) \mid k = 0, \dots, b-1\}$.
2. For any $\lambda \in \sigma(T) \cap \mathbb{T}$, λ is a nondegenerate eigenvalue of T .
3. There exists a unitary matrix $U \in M_n$ such that

$$T(U^k) = \exp\left(\frac{2\pi i}{b}k\right)U^k, \quad k = 0, \dots, b-1.$$

4. The unitary matrix U in 3 has a spectral decomposition

$$U = \sum_{k=0}^{b-1} \exp\left(\frac{2\pi i}{b}k\right)P_k,$$

with spectral projections satisfying

$$T(P_k) = P_{k-1}, \quad \text{mod } b.$$

5. The restriction $T^b|_{P_k M_n P_k}$ of T^b on $P_k M_n P_k$ defines a primitive unital CP map on $P_k M_n P_k$.
6. There exists a faithful T -invariant state φ .

Proof. See [W], for example. \square

E A useful fact on ${}_l C_k$.

Lemma E.1. *Let $l \in \mathbb{N}$, $m_1, m_2 \in \mathbb{N} \cup \{0\}$ such that $m_1 + m_2 \leq l$. Then we have*

$${}_l C_{m_1} \cdot {}_l C_{m_2} = \sum_{k=0}^{m_1+m_2} \alpha_{(m_1, m_2)}^{(k)} \cdot {}_l C_k,$$

where

$$\alpha_{(m_1, m_2)}^{(k)} = \begin{cases} {}_k C_{m_2} \cdot \sum_{j=0}^{m_2} \delta_{m_1, k-j} \cdot {}_{m_2} C_j, & \text{if } k \geq m_2 \\ 0, & \text{if } k < m_2 \end{cases}.$$

Proof. This can be checked inductively. □

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